Index Tracking With Moment Conditions

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Abstract

Tracking error minimization is a widely used by traditional index or passive fund managers for the purpose of gauging how well a given portfolio follows an index. We propose the method of moments to match the index more accurately with a smaller number of assets. The ultimate goal is to take account of higher order moment conditions such as skewness and kurtosis in the return distribution. We discuss how this method compare to each other as well as any improvement over the standard tracking error it is able to yield.

1 Introduction

Index fund mangers evaluate the performance of their portfolio by minimizing the tracking error. The tracking error is a standard means of ascribing a numerical value that shows how well a given portfolio matches a certain financial index. There seem to be a number of definitions of this tracking error in the literature; however, for our purposes we define it as the standard deviation of variance of the index returns minus the portfolio returns. The issue with this formulation of the tracking error is that since it focuses on matching what is essentially the variance of both the portfolio and index, higher order moments such as skewness and kurtosis are not accounted for. Both skewness and kurtosis are relevant since these moments are characterizing outliers and the tails of the distribution. The failure to capture these higher order moments clearly shows the shortcoming of the standard tracking error. If return distributions were normally distributed this would not be an issue since a normal distribution is completely summarized with mean and variance. However it is been shown in numerous papers that returns do not follow normal distributions(Egan, 2007). This means we must seek alternative methods in order to better characterize these return distributions and ultimately match the portfolio return distribution to the index it is attempting to replicate.

Besides mean and variance, no prior work examines the effectiveness of higher order moments in the return distribution. Therefore, we use method of moments to formulate characteristics of a distribution including mean, variance, skewness, and kurtosis in order to examine the effect of higher order moments. Our method of moments is a part of the generalized method of moment but here We follow the just identification condition in the generalized method of moments. When the model is just identified, we use the same number of assets as the number of our moment conditions. Hence, we select four assets from the index to track the index under four moment conditions. Just identification condition can benefit portfolio managers as fewer assets avoids very small and illiquid positions in the portfolio and also prevents transaction and administration costs.

In our experiments, we start with proving the equal effectiveness of tracking error and first two moment conditions and proceed with two different types of distributions: normal distribution and Pearson random distribution using higher order moments. We analyze two moment conditions and four moment conditions under the normal distribution. Under Pearson distribution, where we adopt mean, standard deviation, skewness, and kurtosis of our sample data, we compare the performance of index tracking with two moment conditions and four moment conditions.

Section 2 reviews the literature regarding various applications of tracking error in index tracking and portfolio optimization. Section 3 elaborates our specified methodologies and experiments and Section 4 shows our empirical finding and results.

2 Literature Review

2.1 Tracking Error

The passive investment strategy becomes pervasive during late last century and many fund investors tends to track a benchmark return by minimizing the sum of the squared deviation of the return of their portfolio from a benchmark. It formed as tracking error volatility and the problem was solved by Roll (1992) using quadratic models. Although the tracking error volatility can be employed to yield many statistical measures but it was not easy to interpret by practitioners. Clarke et al (1994) defines the tracking error as the absolute difference between returns of an investor's portfolio and a benchmark return. Therefore, minimizing tracking error becomes a linear tracking error model which can be solved using linear programming approximation. Rudolf, Wolter, and Zimmermann (1998) later show that the linear tracking error models are consistent with expected utility maximization. A simple heuristic algorithm as described in Jansen and Dijk (2002) is also widely used for solving the cardinality-constrained index tracking problems. It utilizes the matrix that subtracts the percentage of asset weights in the benchmark from the percentage of portfolio invested in assets by investors and the covariance matrix of the asset returns.

$$TE = \sqrt{VAR(r_P - r_B)} = \sqrt{E[(r_P - r_B)^2] + (E[r_P - r_B])^2} = \sqrt{(w_P - w_B)^T Q (w_P - w_B)}$$

where r_P and r_B are the portfolio return and benchmark return respectively and w_P, w_B, and Q are the weight of assets in investor's portfolio, weight of assets in the benchmark, and the covariance matrix of asset returns.

However, the literature only defines the mean and variance in the tracking error and uses it as the measure of the performance of passive investment funds. It has been proved by Egan(2007) that SP index returns do not follow a simple normal distribution. Considering the significance of tails and outliers in the return distribution, we, therefore, propose that other factors in the distribution such as skewness and kurtosis should be measured and we solve this problem by utilizing method of moments to take higher order moments into consideration when tracking a benchmark.

2.2 Method of Moments

Method of moments and generalized method of moments have been applied to many empirical research in finance, especially in the area of asset pricing as it takes more realistic assumptions regarding the stochastic process. Before the application of generalized method of moments, the most common econometric tool in the asset-pricing area is the maximum likelihood method. However, the maximum likelihood method is exposed to limitations such as linear approximation and distribution assumptions. Generalized method of moments overcomes those limitations by using variables that can be serially correlated and conditionally heteroscedastic and examining non-linear asset-pricing model without linearization. Nonetheless, no prior literature in field of index tracking uses generalized method of moments given its characteristics of accommodating different distributions. Since index returns are mostly non-normally distributed, generalized method of moments should be able to capture more elements in the distribution and help practitioners track an index more precisely.

3 Methodology

3.1 Data

We use Dow Jones Industrial daily close price data of all 30 assets and the benchmark weight assigned to each asset from March 20 2019 to May 15 2020. We calculate the return, mean, variance, skewness, and kurtosis of the data and simulate daily 40 asset returns under normal distribution with given mean and variance. The simulation has also been done under Pearson random distribution with given mean, variance, skewness, and kurtosis.[1] The purpose is to demonstrate different effectiveness of higher order moments in normally and non-normally distributed return data and how our methods could perform better under non-normally distributed return data.

3.2 Method of Moments

We construct three moment matrices: first with the traditional tracking error, second with two moments only, third with four moments including mean, variance, skewness, and kurtosis, and the third with two moments. Our ultimate goal is to find a portfolio weight allocation that can minimize our moment vectors within matrices. Therefore, we normalize the moment vectors to a scalar value and attempt to minimize the normalized vector within our optimizer.

We formulate in the optimizer to minimize our normalized vector with the constraint that the weight on each asset has to add up to be one and it is also non-negative since we only consider mutual index fund with no leverage.

$$minf(x)\begin{cases} A * \mathrm{x} \leq \mathrm{b} \\ Aeq * \mathrm{x} = beq \end{cases} \tag{1}$$

where A and Aeq are matrices, and b and beq are vectors.

[1]The correlation is not adjusted due to time constraint

3.3 Experiments

In our first experiment, under Pearson random distribution, we compare the traditional tracking error with first two moment conditions, mean and variance, to demonstrate that moment conditions can serve the same rule as the traditional tracking error.

Our second experiment simulates asset data under normal distribution and identify the effectiveness of two and four moment conditions. We conduct this experiment by applying our simulated data under normal distribution and using two normalized vectors, one with mean and variance and the other with mean, variance, skewness, and kurtosis.

Our third experiment introduces Pearson distribution which simulates data with targeted mean, variance, skewness, and kurtosis. We apply two and four moment conditions to track the index with the data simulated from Pearson distribution and address the significance of how higher order moments could track the index better with fewer outliers.

In our judging criteria, we use the tracking error of different experiment from 1000 simulations and conduct t-test on the mean and equal variance test. We also look at the box plot of the tracking errors from different experiments. However, it is a limitation that we consider more factors such as skewness and kurtosis in our distribution but only use tracking error as our judging criteria.

4 Result

In the first experiment of comparing the traditional tracking error with two moment conditions under Pearson random distribution, we find out that the p-value of the t-test of two tracking error samples is smaller than 0.05 , which suggests that two sample means are significantly different. The h value of the equal variance test is 1, which means that two sample variances are significantly different.

In the second experiment of comparing the two moment conditions with four moment conditions under normal distribution, we find out that the p-value of the t-test of two tracking error sample is smaller than 0.05, which suggests that two sample means are significantly different. The h value of the equal variance test is 1, which means that two sample variances are significantly different.

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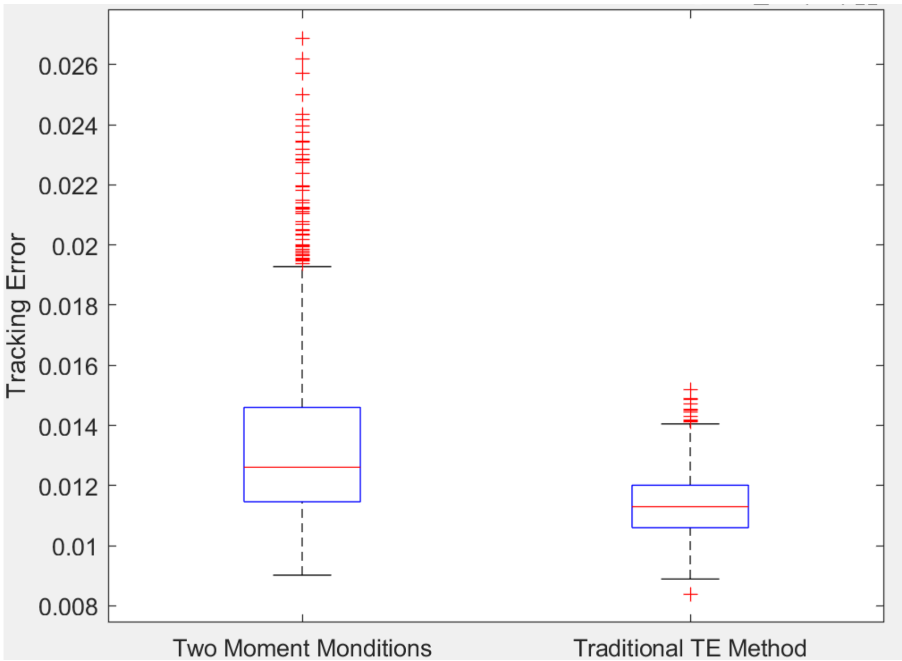

Figure 1: First Experiment

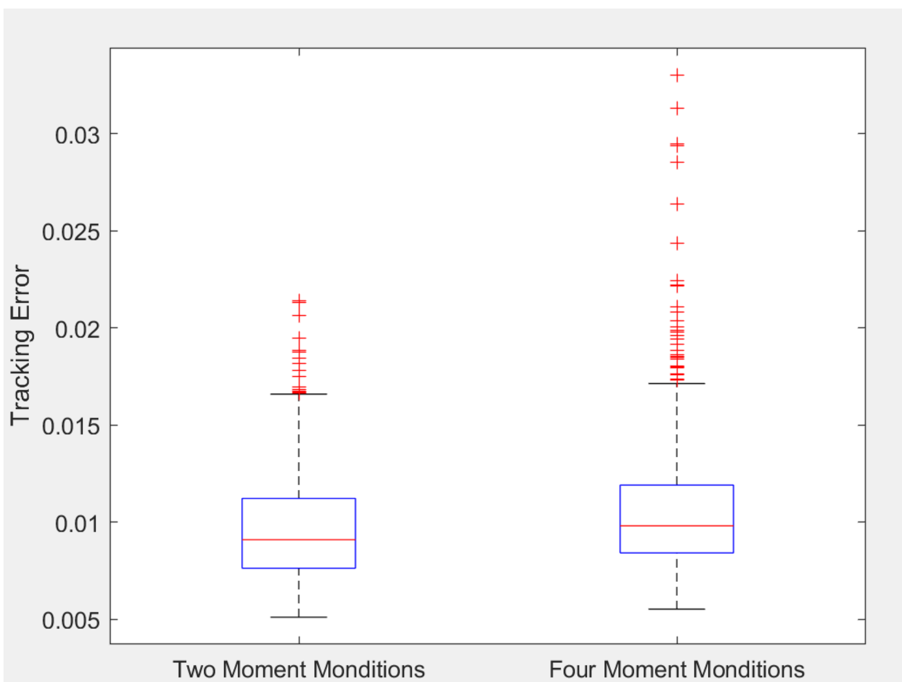

Figure 2: Second Experiment

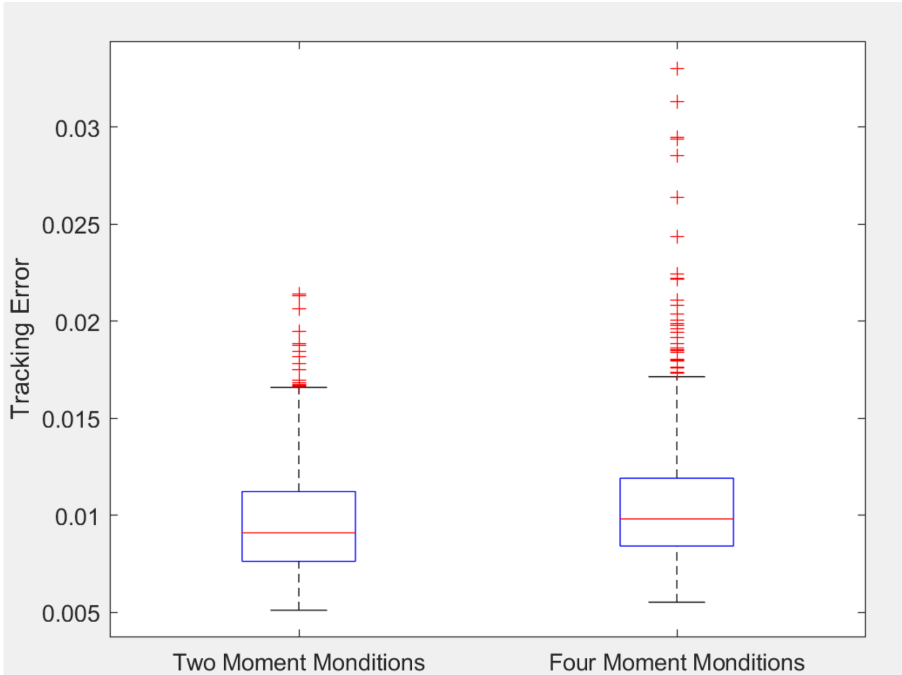

Figure 3: Third Experiment

5 Conclusion

From the result of our experiments, the tracking error distributions from optimizing traditional tracking error and optimizing first two order moments, mean and variance, are different in terms of their mean and variance. The tracking error distributions from optimizing first two moments and four moments, mean, variance, skewness, and kurtosis, with normal distributed data are different in terms of their mean and variance. The tracking error distributions from optimizing first two moments and four moments, mean, variance, skewness, and kurtosis, with Pearson random distributed data are also different in terms of their mean and variance.

The result does not seem to meet our hypothesis that we could use moment conditions to replace tracking error and that the higher order moments could do a better job capturing outliers in the return distribution. However, the result attributes to the fact that we only use the tracking error as our judging criteria which narrows down the higher order of moments we try to measure. A better method of comparing distributions should be employed in order to identify the significance of applying higher order moments. Ultimately we want to develop a method that could optimize the index tracking process by accounting for tails and outliers in the distribution. After we develop the method, a more detailed analysis could be done to minimize the amount of assets that index fund managers use to track a certain benchmark without losing any characteristics of the index return distribution.

6 Reference

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