Online Appendix: Signing-out Confounding Shocks in Variance-Maximizing Identification Methods

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1. Confounding nature of shocks in a simple New Keynesian model

In the New Keynesian model outlined in section 1 of the paper, the solution to the path of the endogenous variables can be written as a function of the structural shocks, η and ϑ. Ψ is a 2×2 matrix which reflects the impact coefficients on the endogenous variables, $\tilde{y}$ and π.

$$\begin{bmatrix}\tilde{y}\\ \pi\end{bmatrix} = \begin{bmatrix}\Psi_{y\eta} & \Psi_{y\vartheta}\\ \Psi_{\pi\eta} & \Psi_{\pi\vartheta}\end{bmatrix}\begin{bmatrix}\eta_t\\ \vartheta_t\end{bmatrix}$$

Where through the method of undetermined coefficients,

$$\Psi_{\pi\eta} = \frac{-\sigma\tilde{\kappa}}{(1-\beta\rho_\eta)(\sigma(1-\rho_\eta)+\phi_y)+\tilde{\kappa}(\phi_\pi-\rho_\eta)},$$

$$\Psi_{y\eta} = \frac{-\sigma(1-\beta\rho_\eta)}{(1-\beta\rho_\eta)(\sigma(1-\rho_\eta)+\phi_y)+\tilde{\kappa}(\phi_\pi-\rho_\eta)},$$

$$\Psi_{\pi\vartheta} = \frac{-\Psi_{NR}\tilde{\kappa}}{(1-\beta\rho_\vartheta)(\sigma(1-\rho_\vartheta)+\phi_y)+\tilde{\kappa}(\phi_\pi-\rho_\vartheta)},$$

$$\Psi_{y\vartheta} = \frac{-(1-\beta\rho_\vartheta)}{(1-\beta\rho_\vartheta)(\sigma(1-\rho_\vartheta)+\phi_y)+\tilde{\kappa}(\phi_\pi-\rho_\vartheta)},$$

In the simulated IRF biases in section 3, the following parameter values are used: $\beta = 0.99$, $\sigma = 1$, $\chi = 1$, $\theta = 0.66$, $\phi_\pi = 1.5$, $\phi_{\tilde{y}} = 0.125$, $\rho_\eta = 0$, and $\rho_\vartheta = 0$. Ψ_{NR} is equal to $-\sigma\frac{1+\chi}{\sigma+\chi}$ and $\tilde{\kappa}$ is equal to $(\sigma+\chi)(1-\theta)(1-\beta\theta)/\theta$.

The structural shocks can be mapped to the reduced form impacts that would be observed by the practitioner, and the variance-maximizing shock determined as a function of these true underlying impulses. We focus on the initial impact period in order to minimize the complexity of the algebra and assume no shock persistence. The reduced-form residuals are

$$\boldsymbol{\varepsilon_t} = \begin{bmatrix}\varepsilon_t^{\tilde{y}}\\ \varepsilon_t^{\pi}\end{bmatrix} = \begin{bmatrix}\Psi_{y\eta} & \Psi_{y\vartheta}\\ \Psi_{\pi\eta} & \Psi_{\pi\vartheta}\end{bmatrix}\begin{bmatrix}\eta_t\\ \vartheta_t\end{bmatrix}$$

Assuming uncorrelated structural shocks with unit variance, and the fact that $E[\eta_t\vartheta_t] = 0$ the variance covariance matrix of residuals is

$$\Sigma = \begin{bmatrix}\Psi_{y\eta}^2+\Psi_{y\vartheta}^2 & \Psi_{y\eta}\Psi_{\pi\eta}+\Psi_{y\vartheta}\Psi_{\pi\vartheta}\\ \Psi_{y\eta}\Psi_{\pi\eta}+\Psi_{y\vartheta}\Psi_{\pi\vartheta} & \Psi_{\pi\eta}^2+\Psi_{\pi\vartheta}^2\end{bmatrix}$$

Let $\tilde{A}$ be the Cholesky decomposition of Σ, using the fact that:

$$\Sigma = \begin{bmatrix}a & 0\\ b & c\end{bmatrix}\begin{bmatrix}a & b\\ 0 & c\end{bmatrix} = \begin{bmatrix}a^2 & ab\\ ab & b^2+c^2\end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix}\sqrt{\Psi_{y\eta}^2+\Psi_{y\vartheta}^2} & 0\\ \frac{\Psi_{y\eta}\Psi_{\pi\eta}+\Psi_{y\vartheta}\Psi_{\pi\vartheta}}{\sqrt{\Psi_{y\eta}^2+\Psi_{y\vartheta}^2}} & \frac{\Psi_{\pi\eta}\Psi_{y\vartheta}-\Psi_{\pi\vartheta}\Psi_{y\eta}}{\sqrt{\Psi_{y\eta}^2+\Psi_{y\vartheta}^2}}\end{bmatrix}$$

$\tilde{A}$, can be combined with the selection matrix ($s = \begin{bmatrix}1 & 0\end{bmatrix}$) to target the output gap, $\tilde{y}$, in order to form the matrix that is used to identify the dominant shock using the eigenvalue-eigenvector approach of Faust (1998).

$$V = \left[\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\Psi_{y\eta}^2 + \Psi_{y\vartheta}^2} & 0 \\ \frac{\Psi_{y\eta}\Psi_{\pi\eta} + \Psi_{y\vartheta}\Psi_{\pi\vartheta}}{\sqrt{\Psi_{y\eta}^2 + \Psi_{y\vartheta}^2}} & \frac{\Psi_{\pi\eta}\Psi_{y\vartheta} - \Psi_{\pi\vartheta}\Psi_{y\eta}}{\sqrt{\Psi_{y\eta}^2 + \Psi_{y\vartheta}^2}} \end{bmatrix} \right]' \left[\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\Psi_{y\eta}^2 + \Psi_{y\vartheta}^2} & 0 \\ \frac{\Psi_{y\eta}\Psi_{\pi\eta} + \Psi_{y\vartheta}\Psi_{\pi\vartheta}}{\sqrt{\Psi_{y\eta}^2 + \Psi_{y\vartheta}^2}} & \frac{\Psi_{\pi\eta}\Psi_{y\vartheta} - \Psi_{\pi a}\Psi_{y\eta}}{\sqrt{\Psi_{y\eta}^2 + \Psi_{y\vartheta}^2}} \end{bmatrix} \right]$$

The eigenvalues of V are the vector $[\Psi_{y\eta}^2 + \Psi_{y\vartheta}^2, 0]$, while the eigenvector corresponding to the largest eigenvalue is $\Gamma_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Notice, that the coefficients from both shocks appear in the eigenvalue, and therefore, the structural rotation in the variance-maximizing approach to identification. The final identification matrix, A^{-1}, is equal to $\tilde{A}\Gamma$, where Γ is a matrix of eigenvectors in descending order following the first column. The first column of Γ is the only shock of interest in this approach. The relative impact of both shocks will determine how close we get to the true structural space (given we have normalized both structural shocks to have unit variance). Notice also that the impact of the dominant shock on inflation will also be increasingly biased in proportion to the size of $\Psi_{y\vartheta}\Psi_{\pi\vartheta}$ relative to $\Psi_{y\eta}\Psi_{\pi\eta}$.

The maximization problem will take a more complex form where the maximization targets dominant shocks over an extended horizon (k) or frequency, in which case:

$$V = \left[\begin{bmatrix} 1 & 0 \end{bmatrix} \sum_{\tau=0}^{k} D^{\tau}\tilde{A} \right]' \left[\begin{bmatrix} 1 & 0 \end{bmatrix} \sum_{\tau=0}^{k} D^{\tau}\tilde{A} \right]$$

The principle is the same however, with the final identified structural shock vulnerable to bias the larger the share of variance driven by the shock that is not of interest.

2. *Constrained variance maximization in larger models*

Sign and relative magnitude constraints can sharpen identification in larger and more complex models than the simple two-variable model given in the main text, and when using a more complicated form for V in the maximization problem. For example, the model of Smets and Wouters (2007) contains 7 shocks. Three of the shocks have characteristics of a typical "demand" shock, in the sense that they generate a positive co-movement between output, prices, and interest rates, and three have "supply" shock characteristics, generating a negative relationship between output relative to inflation and interest rates. The model also contains an additional monetary policy shock.[1]

Seven variables are included in the VAR, based on 1000 periods of simulated data from the Smets-Wouters model: output, hours, wages, investment, consumption, interest rates, and inflation. In the unconstrained case, the shock that dominates business-cycle variation in output is highly persistent; the positive boost of output is accompanied by initial reductions in prices and interest rates (Figure 1). It is also clearly a hybrid shock, as in the previous example, capturing elements of both the demand and supply shocks shown in the blue and red lines respectively, although the latter appear dominant. In the Smets-Wouters model, the supply-side shocks marginally dominate the business-cycle frequency variance of GDP, accounting for around 60 percent.

Sign and elasticity constraints are then applied using our knowledge of the model. The Phillips curve is very flat in the model and the output-inflation relationship is estimated to be just 0.05. This minimum relative magnitude is imposed on impact to isolate demand-drivers of output. To isolate the supply-type drivers, a negative inflation-to-output relative magnitude is imposed. As some of the supply-type drivers do not affect output on impact, the positive output restriction is imposed after one

[1]The "demand"-type shocks are an exogenous spending shock, a risk premia shock, and an investment-specific technology shock. The "supply" shocks reflect neutral technology shocks, a price markup, and a wage markup shock.

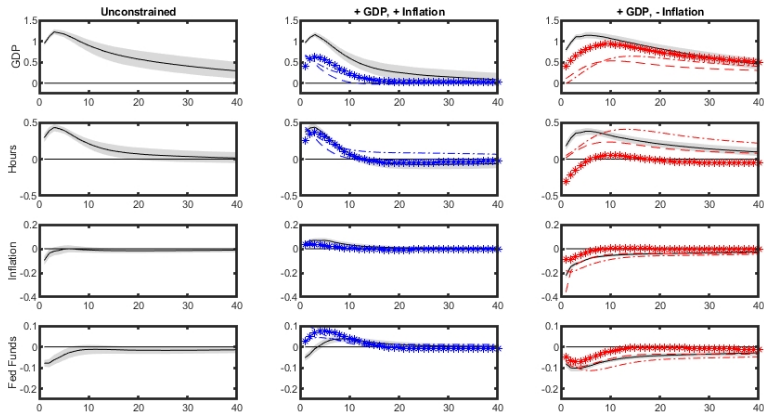

Figure 1. : **Dominant driver of output in Medium-Scale New Keynesian model**

Note: Note: Identified dominant driver of GDP in simulated data produced by the Smets-Wouters model. Blue lines show IRFs from "demand" type shocks in the model, red lines show shocks from supply-type shocks in the model.

year. The smallest inflation to output relative magnitude of the supply drivers in the model is 0.1, so this is imposed as a minimum constraint.

In each restricted identification, the impact response of output is larger than any one of the model's demand (blue) or supply (red) shocks (Figure 1). This reflects the identification capturing a composite combination of the shocks of interest. For the positive and negative relative magnitude restricted cases, the impact on GDP is smaller than in the unconstrained case, capturing a smaller subset of shocks, and yields either less (demand) or more persistent (supply) IRFs than the unconstrained case. Second, the restricted shocks more closely match the shocks of interest than in the unrestricted case, even where no direct elasticity restriction is applied. For example, the response of interest rates is more positive in the positive inflation restriction case, and more negative in the negative inflation restriction case than in the unrestricted case. The response of hours is less persistent in the positive inflation restricted case than the unrestricted case, more closely matching the underlying demand shocks.

3. *Data and complete IRFs in US VAR*

The VAR estimated on U.S. data contains 8 variables, constructed from the Reserve Bank of St. Louis's FRED database (Table 1):

Table 1—: **US data and FRED codes or source**

Variable	Code and transformation
GDP per capita	log(A939RX0Q048SBEA)*100
Utilization-adj. TFP	log(cumsum(Util-adj TFP/4))*100 Fernald (2014)
Hours per capita	log(PRS85006023*(CE16OV/CivPop))*100
Unemployment rate	UNRATE
Investment share of GDP	log(100*((PCDG+GPDI)./GDP))*100
Consumption share of GDP	log(100*((PCND+PCESV)./GDP))*100
Inflation	(log(DPCERD3Q086SBEA)-log(lag(DPCERD3Q086SBEA))*100;
Interest rates	FEDFUNDS

Only a subset of IRFs are shown in the empirical results of the main text to conserve space. All IRFs under the three identifications are shown below in Figure 2

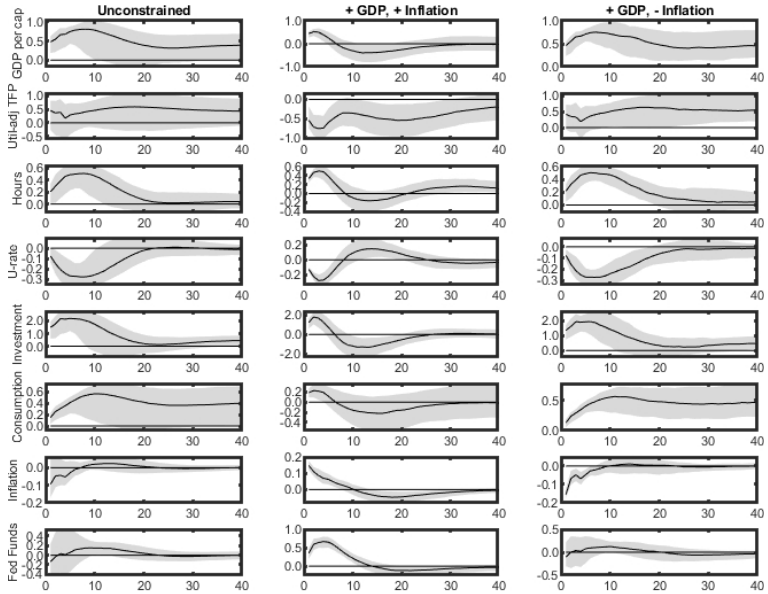

Figure 2. : **Targeting output at business-cycle frequencies, constrained and unconstrained: U.S. Data**

Note: 16th and 84th percentile error bands. Columns reflect the unconstrained eigenvalue-eigenvector solution to the shock which maximizes business-cycle frequency variation of real GDP; the maximizing shock where the impact on GDP is constrained to be positive, and the impact on inflation is at least 0.3 times the GDP impact; and, the maximizing shock where the impact is constrained to be positive for GDP, but at least -0.3 times the GDP impact for inflation.

4. Employment-targeting shock

In Angeletos, Collard and Dellas (2020), the dominant driver of the unemployment rate at business cycle frequencies is the focus of much of the paper, rather than GDP. Targeting unemployment rather than GDP yields IRFs which are more consistent with a "demand"-type shock in the unconstrained case. This may be because demand shocks drive a larger proportion of business-cycle frequency variation in unemployment than GDP. For example, in the Smets-Wouters model, the three demand shocks in the model account for about 60 percent of the business cycle variation of unemployment, but just 40 percent of the variation of GDP.

When targeting the shock that maximizes the business-cycle frequency variation of unemployment, some differences and similarities emerge relative to the shock targeting GDP (Figure 3). First, the unrestricted shock is inflationary, rather than initially deflationary, likely reflecting the increased importance of demand-drivers in unemployment. Second, the shock where the ratio of the response of inflation to GDP is constrained to be positive is also less persistent for GDP than in the unrestricted case, while the negative restriction results in a more persistent impact. This second result is also common to the GDP-targeting maximization.

5. Do dominant business-cycle shocks also explain low-frequency variation of macroeconomic variables?

Angeletos, Collard and Dellas (2020), using a main-business cycle shock targeting unemployment (6-32q), find a disconnect between the dominant business-cycle and long-run drivers of the macroeconomy. Using the same VAR provided in the main text, it is also found that the unconstrained unemployment-targeting business-cycle shock explains 42 percent of business-cycle variation of GDP per capita, but just 11 percent of long-run variation (40+ quarters). However, we find this to be largely the result of the contamination of the main business-cycle driver of unemployment with both demand and supply side drivers.

In contrast, when we apply our methodology of including additional constraints in the maximization problem, we find that supply-side components of the main business cycle shock driving unem-

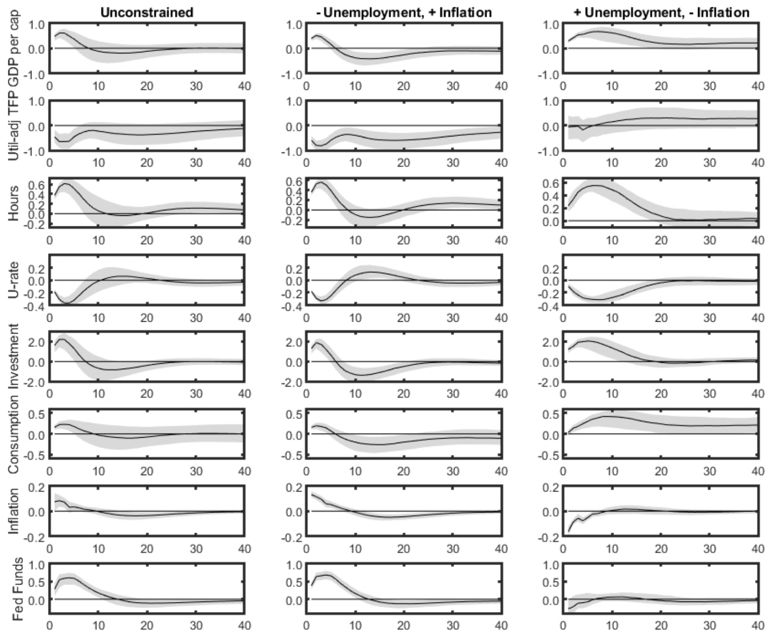

Figure 3. : **Targeting unemployment at business-cycle frequencies, constrained and unconstrained: U.S. Data**

Note: 16th and 84th percentile error bands. Columns reflect the unconstrained eigenvalue-eigenvector solution to the shock which maximizes business-cycle frequency variation of unemployment; the maximizing shock for unemployment where the impact on GDP is constrained to be positive, and the impact on inflation is at least 0.3 times the GDP impact; and, the maximizing shock for unemployment where the impact is constrained to be positive for GDP, but at least -0.3 times the GDP impact for inflation.

ployment explain over 25 percent of both the business-cycle and long run variation of GDP for a sufficiently large restriction on the inflation response relative to GDP. It is the demand components of the shock which do not explain low frequency variation in GDP per capita, while driving a large share of business-cycle variation (Table 2).

6. *Sign and magnitude restrictions without variance maximization*

As we note in the main text, simple sign and magnitude restrictions without the conditioning assumption of an objective maximization problem are simply drawn from a uniform distribution (Haar prior). They therefore can provide very different IRFs than our constrained maximization identification. Figure 4 shows the results of applying the same magnitude restrictions (demand: inflation must rise by 0.3 times the impact on output, supply: inflation must fall by 0.3 times the impact on output). The sign restrictions-only estimations mainly deliver uninformative impulse responses that are indistinguishable from zero.

Table 2—: **Maximizing business cycle variation in unemployment: share of variance of GDP explained at business and long-run frequencies.**

Business cycle (6-32q)					
Scale of restriction	**0.05**	**0.1**	**0.2**	**0.3**	**0.4**
Positive inflation	40	40	39	38	38
	(32, 50)	(32, 50)	(30, 48)	(29, 48)	(28, 48)
Negative inflation	34	34	34	35	36
	(26, 44)	(24, 44)	(22, 44)	(20, 47)	(22, 47)
Long-run (40+q)					
Scale of restriction	**0.05**	**0.1**	**0.2**	**0.3**	**0.4**
Positive inflation	10	9	8	9	12
	(4, 22)	(4, 20)	(3, 20)	(3, 22)	(4, 26)
Negative inflation	18	22	26	31	30
	(8, 33)	(9, 38)	(11, 44)	(13, 49)	(14, 50)

Note: The median percent contribution of the dominant driver of business-cycle frequency variation in unemployment to business-cycle and low-frequency variation in GDP as the restriction on the scale of the inflation response to the identified shock relative to the GDP response is altered. 16th and 84th percentiles shown in brackets.

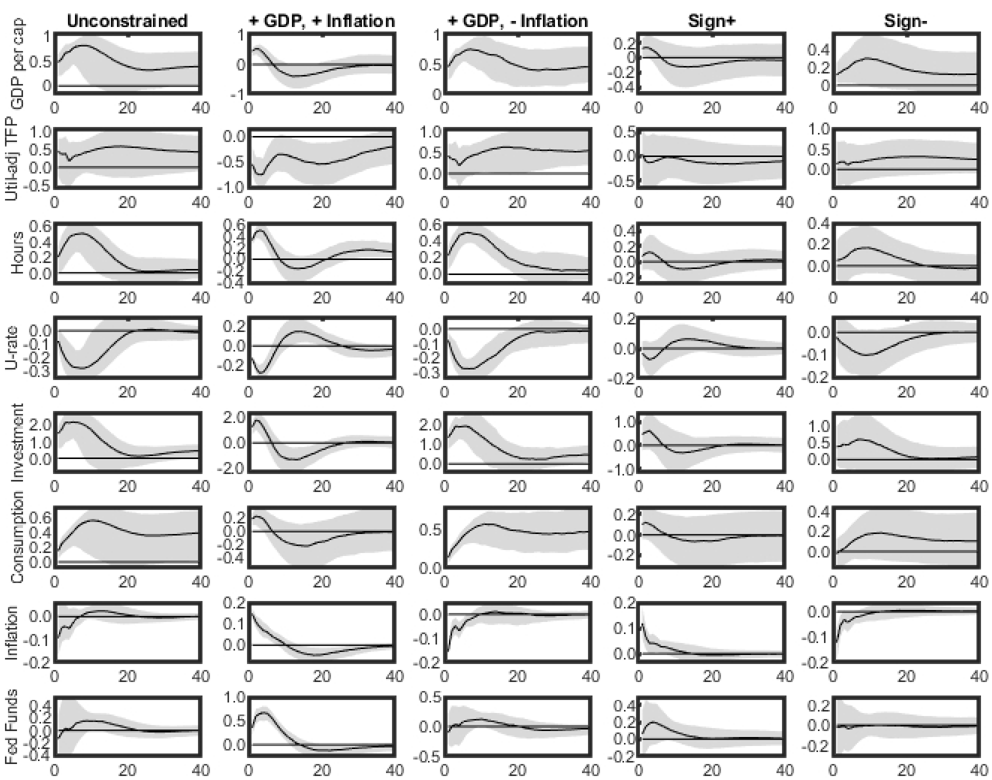

Figure 4. : **Targeting output at business-cycle frequencies, constrained and unconstrained maximization compared to sign and magnitude restrictions: U.S. Data**

Note: 16th and 84th percentile error bands. Columns reflect the unconstrained eigenvalue-eigenvector solution to the shock which maximizes business-cycle frequency variation of real GDP; the maximizing shock where the impact on GDP is constrained to be positive, and the impact on inflation is at least 0.3 times the GDP impact; and, the maximizing shock where the impact is constrained to be positive for GDP, but at least -0.3 times the GDP impact for inflation. Final two columns reflect only sign and magnitude restrictions that constrain the inflation response to be +0.3 and -0.3 of the GDP impact, respectively.

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