AFFINE FRACTIONAL STOCHASTIC VOLATILITY MODELS

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Abstract. By fractional integration of a square root volatility process, we propose in this paper a long memory extension of the Heston (1993) option pricing model. Long memory in the volatility process allows us to explain some option pricing puzzles as steep volatility smiles in long term options and co-movements between implied and realized volatility. Moreover, we take advantage of the analytical tractability of affine diffusion models to clearly disentangle long term components and short term variations in the term structure of volatility smiles. In addition, we provide a recursive algorithm of discretization of fractional integrals in order to be able to implement a method of moments based estimation procedure from the high frequency observation of realized volatilities.

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1. Introduction

The empirical option pricing literature, which is based on the Black and Scholes (1973)(BS) model, has focused heavily on problems surrounding the measurement of the volatility parameter σ. Hull and White (1987)(HW) pioneered the use of the continuous-time stochastic volatility models to capture the effect of stochastic variations in this parameter. Renault and Touzi (1996) have shown that a HW model with a stochastic volatility process $\sigma(t)$ independent from the standardized Brownian innovation of the stock price process implies a U-shaped symmetric volatility curve, whereby the volatility $\sigma_{\mathrm{imp,h}}(t)$ extracted at time t from the BS option pricing formula given the observed (HW) option price (for a given time to maturity h) is graphed against the log-moneyness of the option.

More generally, the empirical biases of the BS model have been dubbed the smile effect in reference to a symmetric implied volatility curve, but numerous distorted smiles in the shape of smirks or frowns are inferred more frequently from market data. Renault (1997) and Garcia, Ghysels, Renault (2003) discuss some extensions of the HW option pricing model which account for the presence of an implied volatility smile, smirk or frown in option data. The basic idea is to explain asymmetric smiles by an instantaneous correlation between returns and volatility in the line of Heston (1993) option pricing model.

Irrespective of its interpretation, the stochastic feature of the volatility process and its instantaneous correlation with the return of the underlying asset appear to be relevant for explaining the variation in strike of the pricing performance of standard (BS) option pricing models. However, the remaining puzzle is the so-called term structure of volatility smiles, that is, the fact that the volatility smile effect appears to be dependent in a systematic way on the maturity structure of options. Actually, Sundaresan (2000) first acknowledges that the volatility smile appears to be stronger in short term options than in long term ones, which is consistent with the stochastic volatility interpretation. When volatility is stochastic, the option price appears to be an expectation of the BS price with respect to the probability distribution of the so-called "integrated volatility" $(1/h)\int_t^{t+h}\sigma^2(u)du$ over the lifetime of the option, indeed only a fraction of it in the Heston model (see Renault and Touzi (1996) for the HW case, Romano and Touzi (1997) for extension to Heston model). In other words, the fixed volatility parameter σ^2 of BS appears to be replaced by the average of the volatility process random path over the lifetime of the option. By a simple application of the law of large numbers to the volatility process (assumed to be stationary and ergodic), one realizes that the effects of the randomness of the volatility should vanish when the time to maturity of the option increases and therefore the volatility smile should be erased.

Sundaresan (2000) is nevertheless right to conclude that the term structure of implied volatilities still appears to have patterns that cannot be so easily reconciled. The main issue to address for reconciling short term and long term observed patterns of the term structure of implied volatilities is twofold:

- On the one hand, stochastic volatility effects appear to be still significant for very long maturity options as documented by Bollerslev and Mikkelsen (1999). The implied level of volatility persistence to account for deep volatility smiles in

long term options is huge in the framework of standard (short memory) models of volatility dynamics like GARCH, or stochastic volatility processes spanned by Markov factors.
- On the other hand, observed prices for very short term option cannot be explained inside the stochastic volatility framework without introducing huge volatility risk premia which would become explosive in longer terms.

There is nowadays a quite general agreement on the idea that jumps components in the return process (and possibly in the volatility process itself) are needed to explain very short term option prices. But the focus of interest of this paper is more the specification of a continuous-time stochastic volatility model which could address option pricing puzzle for longer maturities without introducing unrealistic volatility behavior, in both short and long term returns. One may always add some jump components to the new option pricing model proposed here to accommodate specific option pricing issues for shorter terms but this is beyond the scope of this paper.

We propose in this paper a continuous time stochastic volatility model with long memory, as in Comte and Renault (1998)(CR). It allows us to endow the volatility process with high persistence in the long run in order to capture the steepness of long term volatility smiles without overincreasing the short run persistence. Actually, as in CR, the volatility process appears to be not much more persistent in the very short run than any standard diffusion volatility process while it is infinitely more persistent in the long run: the autocovariance function of the volatility process decreases at an hyperbolic rate for infinitely large lags instead of the standard exponential rate.

The main contribution of this paper is to propose a long memory specification of the volatility process and an associated option pricing model which, while maintaining the appealing features of the CR model, appears to be much more tractable for financial applications, including not only derivative asset pricing but also portfolio management. While CR started from the standard log-normal volatility process ($\log \sigma^2(t)$ represented as an Ornstein-Uhlenbeck process) and proposed a fractional integration of it to introduce long range dependence, we specify here the volatility process $\sigma^2(t)$ as an affine process and then perform a fractional integration of it. Of course, log-normal and affine models are the two cornerstones of modern finance to describe positive processes since they are essentially the only positive stationary diffusion processes for which a simple expression for the transition probabilities is known. However, we are going to argue here that the long memory version of the affine volatility process is well suited for option pricing and hedging.

While the square root process, as first studied by Feller (1951) was popularized for interest rates modelling by Cox, Ingersoll and Ross (1985) and generalized to the class of affine processes by Duffie, Pan and Singleton (2000), its relevance for volatility modelling in the context of quadratic-variation-based strategies of portfolio insurance was put forward by Geman and Yor (1993). As stressed by these authors, a first advantage of the square root process is its interpretation as a Bessel squared process, since the class of laws of squares of Bessel processes is stable by convolution operation, a property which is not shared by the commonly used models of positive processes, like the log-normal one. This may justify to see the volatility of the return on a portfolio as a square root process since it is consistent with a similar assumption for the volatility process of its components. This nice aggregation property will be maintained for the long memory extension we propose in this paper.

However, the main reason why the long memory square root (LMSR) volatility process is better suited for practical option pricing and hedging than the CR model (while it shares the same nice theoretical properties) is its relation with the standard practice of computing option prices and associated Greeks from some adaptations of BS formulas applied to BS implied volatilities. Actually, we are going to show that when the volatility process is LMSR, the integrated variance (for a given time to maturity) is endowed with this long memory feature only through its conditional expectation, while higher order conditional centered moments are short memory. This result is important because it means that the model captures the two stylized facts of long memory dynamics of implied volatilities and deep volatility smiles in long term options without introducing some unpalatable long range dependence in the stochastic process of the volatility smiles. In other words, only one long memory state variable (which is well proxied by at the money BS implied volatilities) is needed to compute option prices in practice; besides, conditionally to this variable, one recovers nice Markovian properties of option prices.

Moreover, the LMSR model is specified by a parsimonious set of parameters which are easy to interpret, in relation to both short and long term behavior of the volatility process, and not so difficult to estimate. In these two respects, the great advantage of the LMSR model is to allow us to derive explicit formula of the various moments of interest as simple functions of these parameters. The price to pay for this simplicity is to work with a process, the square root one, the diffusion coefficient of which is not Lipschitz with respect to the state. This leads us to set in this paper a self-contained theory of fractional integration of the square root process which is not a direct application of CR or of the more comprehensive theory which has been developed in the last five years (Alos, Mazet and Nualart (2000), Carmona and Coutin (2000), Hu, Oksendal and Sulem (2003)). However, the discretization scheme we propose for practical implementation of this continuous time model with discrete time data heavily rests upon the recent works of Carmona, Coutin and Montseny (2000) and Coutin and Pontier (2007).

The paper is organized as follows. The general Markov setting is presented in section 2, as well as the relevant pieces of fractional integration theory. This allows us to propose a continuous time stochastic volatility model where the LMSR volatility process is defined, up to an additive constant, as fractional integration of a square root process. The resulting features of volatility persistence are illustrated by a characterization of the autocorrelation function of the volatility process. Section 3 is devoted to the study of quadratic variation as a tool to estimate both integrated and instantaneous variance from high frequency data. Results of Section 4 are the main motivation of this paper. While studying returns and option prices at horizon h, they put a special emphasis on long horizons and also on long range dependence in realized and implied variance. The issue of statistical inference is addressed in Section 5, starting with the problem of the discretization of fractional operators both for simulation and estimation purpose. Then it is shown that all the volatility model parameters can be consistently estimated by simple methods of moments completed with a step of fractional derivation. A short scale Monte Carlo study shows the feasibility of the procedure and discusses the choice of some discretization parameters. Finally, some concluding remarks are given in Section 6. Section 7 gathers the proofs of all previous sections.

2. The general framework

2.1. **Short memory affine volatility process.** Following Heston (1993), a continuous time stock price process $(S(t))$ is endowed with a one-factor square-root stochastic volatility process $\sigma^2(t) = X(t)$ specified by the following diffusion equations:

$$\begin{cases} dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW^S(t) \\ dX(t) = k(\varpi - X(t))dt + \gamma\sqrt{X(t)}dW^\sigma(t). \end{cases} \tag{2.1}$$

Similarly to the Ornstein-Uhlenbeck-like volatility model of Barndorff-Nielsen and Shephard (2001), the great advantage of the square-root process is to display simple linear dynamics while ensuring positivity under the maintained parameter restriction

$$k > 0, \quad \varpi > 0, \quad k\varpi \geq \frac{\gamma^2}{2}. \tag{2.2}$$

More precisely, it can be shown (see e.g. Lamberton and Lapeyre (1996)) that a square-root process $X(t)$ like (2.1) starting from a positive value has a zero-probability to hit the barrier zero within a finite time. Then the spot variance process $\sigma^2(t) = X(t)$ is by (2.1) a stationary linear AR(1) process with:

$$\begin{cases} \mathbb{E}(\sigma^2(t)) = \varpi \\ \mathrm{Var}(\sigma^2(t)) = \frac{\gamma^2\varpi}{2k} \\ \mathrm{cov}(\sigma^2(t+h), \sigma^2(t)) = \frac{\gamma^2\varpi}{2k}e^{-k|h|}. \end{cases} \tag{2.3}$$

The main drawback of the square-root process is to establish a tight connection between mean and variance through the parameter ϖ. This motivates the extension to the affine class of processes as studied by Duffie, Pan and Singleton (2000)

$$\sigma^2(t) = \theta + X(t) \tag{2.4}$$

with $X(t)$ defined by (2.1).

2.2. **Affine volatility process with long range dependence.** We propose to further extend the affine volatility model (2.4) by assuming from now on that, for some fractional integration exponent α in the interval $]0, 1/2[$:

$$\sigma^2(t) = \theta + X^{(\alpha)}(t), \tag{2.5}$$

where $X^{(\alpha)}(t)$ is formally defined by the fractional integral:

$$X^{(\alpha)}(t) = \int_{-\infty}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} X(s)ds. \tag{2.6}$$

The specification (2.6) is an abuse of notation which must be understood in the following way. For any $a > 0$, the positive process $\int_{-a}^{t}[(t-s)^{\alpha-1}/\Gamma(\alpha)]X(s)ds$ is decomposed as

$$\int_{-a}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} X(s)ds = \int_{-a}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (X(s)-\varpi)ds + \frac{(t+a)^{\alpha}}{\Gamma(\alpha+1)}\varpi. \tag{2.7}$$

The $\mathbb{L}^2$-limit when $a \to +\infty$ of the first term of the right-hand-side in (2.7) is the mean-square stationary process:

$$Y(t) = \int_{-\infty}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (X(s)-\varpi)ds. \tag{2.8}$$

More precisely, wa can prove:

Proposition 2.1. *For $0 < \alpha < \frac{1}{2}$, under assumptions (2.1)-(2.2), $Y(t)$ is mean-square stationary and: for $t \to +\infty$:*

$$\left\| Y(t) - \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [X(s)-\varpi]ds \right\|_2 = O\left(t^{\alpha-1/2}\right).$$

Therefore, understanding the specification (2.5), (2.6) through the decomposition (2.7) for a arbitrary large allows us to see the volatility process $\sigma^2(t)$ as mean-square stationary. The reason why we do not define $\sigma^2(t)$ directly through the stationary process $Y(t)$ is that non-negativity of $Y(t)+\theta$ cannot be guaranteed, irrespective of the value of θ. For all practical purpose, the covariance function of the mean-square stationary process $Y(t)$ can be interpreted as the covariance function of $\sigma^2(t)$, for t sufficiently large. By abuse of notations, the two objects will be confused throughout.

Proposition 2.1 stresses the key role of the assumption $\alpha < 1/2$ to ensure mean-square stationarity. However, it is worth realizing that going from $X(t)$ to $X^{(\alpha)}(t)$ is akin to some integration.

Formally, plugging $\alpha = 1$ in Proposition 2.1 would allow to see $X(t)$ as $dX^{(1)}(t)/dt$. Conversely, a formal integration by part on the truncated version

$$\tilde{X}^{(\alpha)}(t) = \int_{-a}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} X(s)ds$$

of $X^{(\alpha)}(t)$ would give for any $a > 0$:

$$\begin{aligned} \int_{-a}^{t} \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} dX(s) &= \left[\frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} X(x)\right]_{-a}^{t} + \int_{-a}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} X(s)ds \\ &= -\frac{(t+a)^{\alpha}}{\Gamma(\alpha+1)} X(-a) + \tilde{X}^{(\alpha)}(t) \end{aligned}$$

and thus

$$\lim_{\alpha \to 0} \tilde{X}^{(\alpha)}(t) = X(t). \tag{2.9}$$

The result (2.9) will warrant throughout the convention of notation:

$$X^{(0)}(t) = X(t).$$

We can then generalize the formula (2.3) for the variance of an affine volatility process:

Proposition 2.2. *Let* $0 \leq \alpha < \frac{1}{2}$,

$$Var(\sigma^2(t)) = \frac{\gamma^2 \varpi}{k^{2\alpha+1}} \frac{\Gamma(1-2\alpha)\Gamma(2\alpha)}{\Gamma(1-\alpha)\Gamma(\alpha)}$$

Note that the variance formula is continuous at $\alpha = 0$ since when $\alpha \to 0$, using that $\Gamma(\alpha) \sim_{\alpha \to 0} (1/\alpha)$, one gets the variance $\gamma^2\varpi/(2k)$ of the square root process. This continuity property is also warranted for the autocovariance function for infinitely short lags:

Proposition 2.3. *For* $0 \leq \alpha < 1/2$ *and* $h > 0$, *in the neighborhood of* $h = 0$:

$$\frac{\operatorname{cov}(\sigma^2(t+h), \sigma^2(t))}{\operatorname{Var}(\sigma^2(t))} = 1 - \frac{(kh)^{2\alpha+1}}{2\alpha(2\alpha+1)\Gamma(2\alpha)} + O(h^2).$$

The above formula for the correlogramm in the neighborhood of $h = 0$ is coherent with the one of the short memory process, that is

$$e^{-kh} = 1 - kh + O(h^2).$$

In other words, for short horizons, the only difference of fractional integration $0 < \alpha < 1/2$ with respect to common short memory ($\alpha = 0$) is to make the correlogramm even smoother for $h > 0$ in the neighborhood of zero:

$$\frac{\operatorname{cov}[\sigma^2(t+h), \sigma^2(t)]}{\operatorname{Var}(\sigma^2(t))} - 1 = hO(h^{2\alpha}).$$

However, the key difference is for large horizon h. While in the case of short memory, correlation decreases with lag at an exponential rate, only an hyperbolic rate is achieved in case of fractional integration:

Proposition 2.4. *For* $0 < \alpha < 1/2$

$$\frac{\operatorname{cov}(\sigma^2(t+h), \sigma^2(t))}{\operatorname{Var}(\sigma^2(t))} \sim \frac{(kh)^{2\alpha-1}}{\Gamma(2\alpha)} \text{ when } h \text{ goes to infinity.}$$

Proposition 2.4 shows that fractional integration allows us to define a volatility process with long memory. Moreover,thanks to the affine structure we can compute the spectral density in closed form:

Proposition 2.5. *For* $0 \leq \alpha \leq 1/2$, *the sepctral density* $f_{\sigma^2}(\lambda)$ *of the mean-square stationary process* $\sigma^2(t)$ *is given for all positive* λ *as:*

$$f_{\sigma^2}(\lambda) = \int e^{-i\lambda h} \operatorname{cov}(\sigma^2(t), \sigma^2(t+h)) dh = \frac{\gamma^2 \varpi}{\lambda^{2\alpha}(\lambda^2 + k^2)}.$$

Proposition 2.5 confirms that all the properties of the covariance function are continuous at $\alpha = 0$ for $\lambda > 0$. By contrast, the long-range behaviour as captured by considering $\lambda \to 0$ is qualitatively different for $\alpha > 0$ and $\alpha = 0$.

3. Quadratic variation

A number of recent papers (see Barndorff-Nielsen and Shephard (2007) and references therein) have put forward nonparametric measurement of volatility through quadratic variation estimation from high frequency data. More precisely, for a diffusion-type model of log-prices $p(t) = \log(S(t))$ as deduced from (2.1)

$$dp(t) = \mu(t)dt + \sigma(t)dW^S(t),$$

the realized volatility

$$\sum_{i=1}^{n}[p(t_i) - p(t_{i-1})]^2$$

over a time grid

$$t = t_0 < t_1 < \cdots < t_n = t + 1$$

converges in mean square when the mesh of the partition $\max_{1\leq i\leq n} |t_i - t_{i-1}|$ goes to zero, towards $\int_t^{t+1} \sigma^2(u)du$.

Moreover, even when "efficient" prices $p(t)$ are observed only up to some microstructure noise, more sophisticated consistent estimators $\hat{V}_{t,t+1}$ of $\int_t^{t+1} \sigma^2(u)du$ can be computed. The rate of convergence and the asymptotic distribution of $[\hat{V}_{t,t+1} - \int_t^{t+1} \sigma^2(u)du]$ are now well documented in the literature for a variety of contexts and estimators.

Then, it is worth noticing that, up to a step of fractional differentiation, our general framework paves also the way for consistent estimation of quadratic variation of the underlying short memory process.

More precisely, following Samko *et al.* (1993, p.99 and 109), we can define for any function f Hölder continuous of some positive order, the fractional derivative

$$f^{(-\alpha)}(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} ds. \tag{3.1}$$

It follows from Proposition 2.1 and Samko (1993), that

Proposition 3.1. *Under Assumption (2.1) and (2.2), for* $0 \leq \alpha < 1/2$, $\sigma^2(t) = \theta + X^{(\alpha)}(t)$ *is Hölder continuous of any order* $\beta < \alpha + 1/2$ *and we have*

$$(\sigma^2)^{(-\alpha)}(t) = X(t) \quad \textit{and} \quad (V_{0,t})^{(-\alpha)} = \int_0^t X(u)du,$$

where $V_{0,t} = \int_0^t \sigma^2(u)du$.

It is worth noting that while long-memory has no impact on the rates of convergence of estimators of integrated variance $\int_0^t \sigma^2(u)du$ on a finite interval, it will impact the estimation of its derivative. We only consider here the case of simple quadratic variation in a context where there is no microstructure noise. We refer to Mykland, Renault and Zhang (2009) for a more general study.

Let us consider the rolling sample estimator

$$\hat{\sigma}^2_{N,p}(t) = \frac{N}{pT} \sum_{k=[tN/T]-p+1}^{[tN/T]} (p(t_k) - p(t_{k-1}))^2, \tag{3.2}$$

with

$$t_k = k\frac{T}{N}, k = \left[\frac{tN}{T}\right], \left[\frac{tN}{T}\right] - 1, \ldots, \left[\frac{tN}{T}\right] - p + 1,$$

where $[z]$ denotes the integer part of the real number z. We are then faced with a standard bias-variance trade off. The larger p, the smaller the variance of $\hat{\sigma}^2_{N,p}(t)$ but the larger its bias with respect to $\sigma^2(t)$ since it is based on more lagged observations. However, thanks to volatility persistence, this is all the less detrimental when α is large:

Proposition 3.2. *Let $p(t) = p(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW^S(s)$ with σ defined as in (2.5) and $\mathbb{E}(|\mu(s)|^4) < M, \forall s \geq 0$. Let T be fixed and N and p such that p/N is small, then*

$$\mathbb{E}[\hat{\sigma}^2_{N,p}(t) - \sigma^2(t)]^2 \leq \frac{2\mathbb{E}(\sigma^4)}{p} + K\left(p\frac{T}{N}\right)^{2\alpha+1} + \frac{K'\sqrt{M\mathbb{E}(\sigma^4)}}{N} + O\left(\left(\frac{pT}{N}\right)^2\right)$$

where K and K' are functions of the parameters of the process $X(t)$ and of α.

Proposition 3.2 extends to the case of affine long memory volatility processes some previous results of Nelson (1991) for short memory processes and Comte and Renault (1998) for log-normal long memory processes.
In the affine setting, the role of the various parameters can be characterized analytically. Moreover, the optimal bias-variance trade off is reached when p goes to infinity as $[N/p]^{2\alpha+1}$ that is $p = [N^{\frac{2\alpha+1}{2\alpha+2}}]$, leading to a rate of convergence

$$N^{-\frac{2\alpha+1}{2\alpha+2}}$$

for the mean squared error. In other words, when the degree of the fractional process increases from 0 to 1/2, p can be increased from $N^{1/2}$ to $N^{2/3}$ with an inversely proportional rate of convergence for the mean squared error.

4. Returns and option prices at horizon h

The term structure of integrated volatilities at time t, that is the function $h \mapsto \int_t^{t+h} \sigma^2(s)ds$ raises some important issues, in particular in relation with option pricing. It is worth reminding that:

First, Black-Scholes implied volatilities at time t for option maturing at time $(t+h)$ are tightly related to the conditional expectation $\mathbb{E}_t\left[(1/h)\int_t^{t+h}\sigma^2(s)ds\right]$ of time-averaged integrated volatility. This so-called expectation hypothesis has been well-documented in empirical option pricing studies including Campa and Chang (1995) and Byoun, Kwok and Park (2003).

Second, the shape of the volatility smile at time t for a given maturity $(t+h)$ must be explained in relation with the conditional probability distribution given $\mathcal{I}(t)$ of the unexpected part

$$U_t(h) = \int_t^{t+h} \sigma^2(s)ds - \mathbb{E}_t\left[\int_t^{t+h} \sigma^2(s)ds\right]$$

of the integrated volatility. In particular, since the steepness of the volatility smile has much to do with a Jensen effect (see Renault and Touzi (1996)), the size of the conditional variance $V_t\left[\int_t^{t+h} \sigma^2(s)ds\right]$ plays a crucial role in its explanation.

Note that the conditional probability distribution of the integrated volatility given $\mathcal{I}(t)$ is actually a conditional probability distribution given the sub-σ-field of the state variables:

$$\mathcal{F}(t) = \sigma\left[X(\tau), \tau \leq t\right].$$

Then, since the volatility process is long memory, one might be afraid that this conditional probability distribution depends upon the whole path $X(\tau), \tau \leq t$ of state variables, making option pricing in this context a daunting task. But the main result of this subsection is that $U_t(h)$ inherits the Markov property of the state variables process $X(t)$. More precisely, we are going to show that the probability distribution of $U_t(h)$ given $\mathcal{F}(t)$ depends upon $\mathcal{F}(t)$ only through the current state $X(t)$. It is actually determined by the conditional probability distribution given $X(t)$ of the path $X(\tau), t \leq \tau \leq t+h$ of the state variables over the lifetime of the option.

This very convenient result is not specific to the affine stochastic volatility model but more generally a consequence of a kind of commutativity property between two integration operators: the fractional integration operator which defines the volatility process on the one hand and the common integration operator which defines the integrated volatility on the other hand. This property allows us to write formally:

$$\begin{aligned}
\int_t^{t+h} \sigma^2(s)ds &= \theta h + \int_{-\infty}^{t+h} X^{(\alpha)}(s)ds - \int_{-\infty}^{t} X^{(\alpha)}(s)ds \\
&= \theta h + X^{(\alpha+1)}(t+h) - X^{(\alpha+1)}(t) \\
&= \theta h + \int_{-\infty}^{t+h} \frac{(t+h-s)^\alpha}{\Gamma(\alpha+1)} X(s)ds - \int_{-\infty}^{t} \frac{(t-s)^\alpha}{\Gamma(\alpha+1)} X(s)ds \\
&= \theta h + \frac{1}{\Gamma(\alpha+1)} \int_{-\infty}^{t} \left[(t+h-s)^\alpha - (t-s)^\alpha\right] X(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha+1)} \int_t^{t+h} (t+h-s)^\alpha X(s)ds.
\end{aligned}$$

Thus the integrated volatility is decomposed in three parts:

$$\int_t^{t+h} \sigma^2(s)ds = \theta h + f_{\alpha,h}(\mathcal{F}_t) + \frac{1}{\Gamma(\alpha+1)} \int_0^h (h-s)^\alpha X(t+s)ds. \tag{4.1}$$

where: θh is its unconditional expectation,

$$f_{\alpha,h}(\mathcal{F}_t) = \frac{1}{\Gamma(\alpha+1)} \int_{-\infty}^{t} \left[(t+h-s)^\alpha - (t-s)^\alpha\right] X(s)ds$$

belongs to the information $\mathcal{F}_t$ available at time t, while, by virtue of the Markovianity of the process $X(t)$, the optimal forecast at time t of the third part is

$$\frac{1}{\Gamma(\alpha+1)} \int_0^h (h-s)^\alpha \mathbb{E}_t X(t+s)ds = g_{\alpha,h}(X(t)) \tag{4.2}$$

for some deterministic function $g_{\alpha,h}$. In the particular case of an Ornstein-Uhlenbeck like state variables process, as for instance the affine model,

$$\mathbb{E}_t X(t+h) = e^{-kh} X(t) \tag{4.3}$$

and then

$$g_{\alpha,h}(X(t)) = G_\alpha(h)X(t) \quad \text{with} \quad G_\alpha(h) = \frac{1}{\Gamma(\alpha+1)} \int_0^h (h-s)^\alpha e^{-ks} ds. \tag{4.4}$$

We are then able to state

Proposition 4.1. *The conditional distribution of*

$$U_t(h) = \int_t^{t+h} \sigma^2(s)ds - \mathbb{E}_t\left[\int_t^{t+h} \sigma^2(s)ds\right] \tag{4.5}$$

given $\mathcal{F}(t)$ is a deterministic function of the conditional probability distribution of $(X(\tau))_{t\le\tau\le t+h}$ given $X(t)$. This deterministic function is defined by

$$U_t(h) = \frac{1}{\Gamma(\alpha+1)} \int_0^h (h-s)^\alpha X(t+s)ds - \frac{X(t)}{\Gamma(\alpha+1)} \int_0^h (h-s)^\alpha e^{-ks} ds.$$

It is worth noting that the linear formula $g_{\alpha,h}(X(t)) = G_\alpha(h)X(t)$ is actually valid for any Ornstein-Ulhenbeck like state variables process X conformable to (4.3). In addition, the affine structure allows us to get closed-form formulas for all higher moments $\mathbb{E}_t[U_t(h)]^k$, $k = 2, \dots$. The nice consequence of the affine structure is that not only these moments depend on $\mathcal{F}(t)$ only through $X(t)$, but this dependence is linear. We make it explicit for $k = 2$:

Proposition 4.2.

$$V_t\left[\int_t^{t+h} \sigma^2(s)ds\right] = A(\alpha,h)X(t) + B(\alpha,h)$$

with

$$A(\alpha,h) = \frac{\gamma^2}{k}\left[\frac{2}{(\alpha+1)\Gamma(\alpha+1)^2}\left(\int_0^h (h-u)^\alpha e^{-ku}(h^{\alpha+1} - (h-u)^{\alpha+1})du\right) - G_\alpha(h)^2\right]$$

and

$$B(\alpha,h) = \frac{\gamma^2\varpi}{2k}\left(\frac{1}{\Gamma(\alpha+1)^2}\int_0^h\int_0^h (h-u)^\alpha(h-v)^\alpha e^{-k|u-v|}dudv - G_\alpha(h)^2\right)$$

and $G_\alpha(h)$ being defined by (4.4).

For h infinitely large, we can derive the following asymptotic equivalents

$$A(h,\alpha) \sim_{h\to+\infty} \frac{\gamma^2 h^{2\alpha}}{k^3\Gamma(\alpha+1)^2} \text{ and } B(h,\alpha) \sim_{h\to+\infty} \frac{\gamma^2\varpi h^{2\alpha+1}}{k^2\Gamma(\alpha+1)^2}.$$

In other words, the conditional variance of the average volatility over the lifetime of the option is in probability equivalent to its deterministic part:

$$V_t\left[\frac{1}{h}\int_t^{t+h}\sigma^2(s)ds\right] \sim_{h\to+\infty} \frac{\gamma^2\varpi h^{2\alpha-1}}{k^2\Gamma(\alpha+1)^2} \tag{4.6}$$

This result is important since it shows that, with a moderate level of long memory in the volatility process, $\alpha = 1/4$ say, this conditional variance is divided by ten when the time to maturity h of the option contract is multiplied by 100. By contrast, the same factor 100 would divide the variance in the short memory case. Since we know (see Renault and Touzi (1996)) that the volatility smile is produced by a Jensen effect due to the randomness of the average volatility $\frac{1}{h}\int_t^{t+h}\sigma^2(s)ds$, the long memory model of volatility dynamics appears much better suited to reproduce observed volatility smiles in long term options (say with a time to maturity larger than six months) than the common short memory models.

Moreover, since we can rewrite (4.6) as:

$$V_t\left[\frac{1}{h}\int_t^{t+h}\sigma^2(s)ds\right] \sim_{h\to+\infty} \frac{\gamma^2\varpi}{k^{2\alpha+1}}\frac{(hk)^{2\alpha-1}}{\Gamma(\alpha+1)^2} \tag{4.7}$$

we can clearly disentangle two effects in the explanation of the volatility smile:

a) The first one, independent of the maturity, is simply produced by the unconditional variance $\gamma^2\varpi/k^{2\alpha+1}$ of the spot volatility process.
b) The second one captures the erosion of the volatility smile when maturity increases. It is given by the term $(hk)^{2\alpha-1}$ where, for a given long memory parameter α, the time to maturity h is scaled by the mean reversion parameter k. The stronger is the mean reversion in the volatility process, the less pronounced the volatility smile will be for long horizons.

On the opposite, we have for infinitely short times to maturity:

$$A(h,\alpha) \sim_{h\to 0} \frac{\gamma^2 h^{2\alpha+3}}{(2\alpha+3)\Gamma(\alpha+2)^2}, \text{ and } B(h,\alpha) \sim_{h\to 0} \frac{\gamma^2\varpi h^{2\alpha+3}}{2(2\alpha+3)\Gamma(\alpha+1)\Gamma(\alpha+3)}. \tag{4.8}$$

Therefore, the conditional variance $V_t\left[(1/h)\int_t^{t+h}\sigma^2(s)ds\right]$ is, when h goes to zero, an infinitely small of order $h^{2\alpha+1}$. In other words, by contrast with the long horizon case, the volatility smile for very short term options will be, ceteris paribus, less pronounced in the case of long memory stochastic volatility than is the common short memory case . This result is consistent with a well documented empirical puzzle about the term structure of implied volatilities. While to explain long term option prices with a short memory level, one would need to introduce a huge level of volatility persistence, this level would be inconsistent with the empirical evidence on very short term options.

About the possible explanations of observed volatility smiles, another warning is in order. It is often claimed that the convexity of the volatility smile is produced by the unconditional excess kurtosis of log-returns. While volatility persistence does produce unconditional kurtosis in general, the study of very short term options precisely shows that volatility persistence and unconditional kurtosis are two distinct phenomena and that the volatility smile does correspond to the first phenomenon.

To see this, let us consider for the sake of notational simplicity that the log-price has a zero deterministic drift and there is no leverage effect, that is the two Wiener processes W^S and W^X are independent. Then, up to an additive constant, the log-returns can be written:

$$R_t(h) = \log \frac{S(t+h)}{S(t)} = \int_t^{t+h} \sigma(u) dW^S(u)$$

and the two processes σ and W^S are independent.

Hence, given the volatility path $\sigma(.)$, the log-return is normal and we can write:

$$\mathbb{E}\left[R_t^2(h)|\sigma\right] = \int_t^{t+h} \sigma^2(u) du$$

and

$$\mathbb{E}\left[R_t^4(h)|\sigma\right] = 3\left(\mathbb{E}\left[R_t^2(h)|\sigma\right]\right)^2 = 3\left(\int_t^{t+h} \sigma^2(u) du\right)^2.$$

We deduce that:

$$\mathbb{E}\left[R_t^2(h)\right] = \int_t^{t+h} \mathbb{E}(\sigma^2(u)) du = h\mathbb{E}(\sigma^2),$$

where $\mathbb{E}(\sigma^2)$ denotes the unconditional expectation of the mean-square stationary volatility process and:

$$\mathbb{E}\left[R_t^4(h)\right] = 3\mathbb{E}\left[\left(\int_t^{t+h} \sigma^2(u) du\right)^2\right] = 3h^2(\mathbb{E}(\sigma^2))^2 + 3 \text{ Var}\left[\int_t^{t+h} \sigma^2(u) du\right].$$

In other words, the unconditional kurtosis of the return over the period $[t, t+h]$ is given by:

$$K(h) = \frac{\mathbb{E}\left[R_t^4(h)\right]}{\left(\mathbb{E}\left[R_t^2(h)\right]\right)^2} = 3\left[1 + \frac{\text{Var}\left[\frac{1}{h}\int_t^{t+h} \sigma^2(u) du\right]}{(\mathbb{E}(\sigma^2))^2}\right]. \tag{4.9}$$

The limit cases $h \to 0$ and $h \to \infty$ are easy to deduce from (4.9):

a) First, since $\frac{1}{h}\int_t^{t+h} \sigma^2(u) du$ converges in mean-square towards $\sigma^2(t)$ when h tends to zero,

$$\lim_{h \to 0} K(h) = 3\left[1 + \frac{\text{Var}(\sigma^2)}{(\mathbb{E}(\sigma^2))^2}\right] \tag{4.10}$$

where $\mathrm{Var}(\sigma^2)$ denotes the unconditional variance of the mean-square stationary spot volatility process. Formula (4.10) is actually an application to very short time intervals of a result well-known since Clark (1973): the excess kurtosis is equal to 3 times the squared coefficient of variation of the stochastic variance. This excess kurtosis effect persists in the very short term even though the volatility smile erases at rate $h^{2\alpha+1}$ as the conditional variance $V_t\left[\frac{1}{h}\int_t^{t+h}\sigma^2(u)du\right]$. This rate is actually also the rate of convergence of the kurtosis coefficient $K(h)$ towards its limit value (4.10).

b) Second, since $\frac{1}{h}\int_t^{t+h}\sigma^2(u)du$ converges in mean-square towards $\mathbb{E}(\sigma^2)$ when h tends to infinity,

$$\lim_{h\to+\infty} K(h) = 3.$$

This limit is reached at the rate $h^{2\alpha-1}$, that is the one which also governs the convergence to zero of the conditional variance $V_t\left[\frac{1}{h}\int_t^{t+h}\sigma^2(u)du\right]$. In other words, volatility smile and excess kurtosis assessments lead to the same conclusion in the very long term. Thanks to long memory, the convergence towards the log-normal case when the time to maturity increases is much slower.

The following proposition states that the kurtosis results described above remain valid in the general case of non-zero drifts and leverage effects. These results are actually deduced from the long memory properties of the volatility process and do not depend on the specific affine structure:

Proposition 4.3. *Let $\breve{K}(h)$ denote the approximate unconditional kurtosis coefficient of the log-return $R_t(h) = \log(S(t+h)/S(t))$ with σ^2 replaced by σ^2. Then*

(i) $K(h) = 3\left[1 + \dfrac{\mathrm{Var}(\sigma^2)}{(\mathbb{E}(\sigma^2))^2}\right] + O(h^{2\alpha+1})$ *when h goes to zero,*

(ii) $K(h) = 3 + O(h^{2\alpha-1})$ *when h goes to infinity.*

Finally, it is worth studying the dynamic properties of the integrated volatility process for a given horizon h, say $h = 1$.

The integrated volatility process is defined as

$$Z(t+1) = \int_t^{t+1}\sigma^2(u)du = \theta + \int_t^{t+1}\left[\int_{-\infty}^{u}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}X(s)ds\right]du.$$

Z(t) is a second order stationary process the spectral density f_Z of which is easily deduced from the spectral density f_X of X:

Proposition 4.4. *Let f_Z and f_X be the spectral density of Z and X respectively. Then*

$$f_Z(\lambda) = \lambda^{-2\alpha} f_X(\lambda)\frac{|e^{i\lambda}-1|^2}{\lambda^2}.$$

In particular for $\lambda \to 0$, $f_Z(\lambda) \sim \lambda^{-2\alpha} f_{\tilde{\sigma}^2}(0)$.

This implies that Z is a stationary long memory process of order α: its autocovariance function at lag h decays to zero with hyperbolic rate $h^{2\alpha-1}$ when h goes to infinity. While simple albeit tedious computations would allow to deduce from Proposition 4.4 the analog of Proposition 2.3 for characterizing the asymptotic behavior of the autocovariance function of the integrated volatility process $Z(t)$, we prefer to focus here on the autocovariance function of the expected integrated volatility process

$$Y(t) = \mathbb{E}_t Z(t+1) = \int_0^1 \mathbb{E}_t \left[\sigma^2(t+u)\right] du.$$

As already announced, we expect that the memory properties of the process $Y(t)$ mimic the ones of Black-Scholes implied volatilities computed on fixed maturities.

It is worth noting that the three processes $\sigma^2(t)$ of spot volatility, $\mathbb{E}_t\sigma^2(t+1)$ of expected spot volatility and $Y(t) = \mathbb{E}_t Z(t+1)$ of expected integrated volatility share the same equivalent of the autocovariance function for infinite lags:

Proposition 4.5. *When $h \to +\infty$ and for $\alpha \in]0, 1/2[$, $cov[\sigma^2(t), \sigma^2(t+h)]$, $cov[\mathbb{E}_t\sigma^2(t+1), \mathbb{E}_{t+h}\sigma^2(t+h+1)]$ and $cov[Y(t), Y(t+h)]$ can all be written*

$$Ch^{2\alpha-1} + o(h^{2\alpha-1}) \quad \textit{with} \quad C = \frac{\varpi\gamma^2}{k^2}\frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)\Gamma(\alpha)}.$$

5. Simulation and estimation

The purpose of this section is twofold. First, we propose some general tools for the various discretization issues which are relevant for the purpose of statistical inference on the affine fractional stochastic volatility model: discretization of the square root diffusion equation, discretization of the fractional integration operator, discretization of the fractional derivation operator. These various discretization schemes may be useful for a variety of inference strategies, including simulation based ones.

We focus in the second part on a specific inference strategy and illustrate it by a short-scale Monte-Carlo study. This strategy is based on continuous record asymptotics of Proposition 3.2 to recover some observations of the spot volatility process. These observations are used to recover the mean volatility and the fractional integration parameter and then, the underlying square root process is estimated by a method of moments.

5.1. Discretization of fractional integrals.

5.1.1. *Discretization of the CIR equation (2.1).* A simple Euler discretization scheme is not well-suited in the case of the CIR equation for two reasons:

- First, the lack of Lipschitz regularity of the square-root diffusion coefficient invalidates standard convergence theory for Euler discretization schemes,
- Second, the positivity requirement will not be met with simulation of normal innovations.

We resort here to a method proposed by Rogers (1995), which is well-founded and user-friendly. Its only drawback is to impose some constraints on the CIR parameters k, ϖ and γ. It can only accommodate parameter values such that $d = 4k\varpi/\gamma^2$ is an integer (see also Diop (2003)) for the general issue of simulating CIR processes). The idea is to start with a d-dimensional Ornstein-Uhlenbeck process:

$$dV_t = -\frac{k}{2} V_t dt + \frac{1}{2}\gamma dW^{(d)}(t) \tag{5.1}$$

where $W^{(d)}(t)$ is a d-dimensional standard Brownian motion. Then

$$X(t) = |V_t|^2 = \sum_{i=1}^{d} V_{i,t}^2$$

satisfies the equation

$$dX(t) = \left[\frac{d\gamma^2}{4} - kX(t)\right] dt + \gamma\sqrt{X(t)} dW^*(t)$$

where W^* is a another standard one-dimensional Brownian motion built on the coordinates of $W^{(d)}$. This gives a CIR equation analogous to the second equation of (2.1) with $\varpi = d\gamma^2/(4k)$ and of course a constraint since d is an integer. Note that we need $d \geq 2$ for the positivity constraint of the process ($k\varpi \geq \gamma^2/2$) to be fulfilled. The stationary solution of equation (5.1) can be written

$$V_t = \int_{-\infty}^{t} e^{-\frac{k}{2}(t-s)} \frac{\gamma}{2} dW^{(d)}(s)$$

and it admits an exact discretization which can be written

$$V_{t+\delta} = e^{-k\delta/2} V_t + \int_{t}^{t+\delta} e^{-\frac{k}{2}(t+\delta-s)} \frac{\gamma}{2} dW^{(d)}(s).$$

In other words, V is a multivariate AR(1) process with

$$V_{(p+1)\delta} = e^{-k\delta/2} V_{p\delta} + \varepsilon_{(p+1)\delta}, \ \ p \in \mathbb{N} \tag{5.2}$$

where the $\varepsilon_{p\delta}$'s are independent identically distributed random variables drawn from a d-dimensional Gaussian $\mathcal{N}(0, \frac{\gamma^2(1-e^{-k\delta})}{4k} I_d)$ where I_d is the identity $d \times d$ matrix, and the initial condition is an independent Gaussian $\mathcal{N}(0, \gamma^2 I_d/(4k))$ variable.

5.1.2. *Discretization of the fractional integration.* We present in this section a discretization scheme for fractional integrals, studied by Carmona, Coutin and Montseny (2000) for Gaussian processes. The aim of all the transformations below are to exhibit a *recursive* discretization method. Indeed, a naive Euler scheme for fractional integrals leads to non-recursive formulas, see Comte and Renault (1998). In that case, the computation of the last term uses systematically all the previous ones, which is computationally very heavy, especially when one wants to deal with large samples.

The method is the result of four consecutive ideas which can be described as follows:

(1) Approximate the operator $X \mapsto X^{(\alpha)}$ by the truncated operator $X \mapsto X_0^{(\alpha)}$ which associates to a process X the process:

$$X_0^{(\alpha)}(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} X(s) ds.$$

By Proposition 2.1, this approximation is consistent when t goes to infinity, with an absolute error of order $t^{\alpha-1/2}$.

(2) Use Laplace inverse transform to write

$$(t-s)^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty x^{-\alpha} e^{-x(t-s)} dx,$$

and apply Fubini's theorem:

$$f_0^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty x^{-\alpha} \left(\int_0^t e^{-x(t-s)} f(s) ds \right) dx.$$

This representation, known as the diffusive representation of fractional integrals, is used by Carmona, Coutin and Montseny (2000) to write that any function f, continuous on $[0,T]$ satisfies:

$$f_0^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty x^{-\alpha} \Psi(x,t,f) dx, \tag{5.3}$$

where $\Psi(x,t,f)$ is the linear operator defined in Coutin and Pontier (2007) for continuous functions on $[0,T]$ by

$$\Psi(x,t,f) = \int_0^t e^{-x(t-s)} f(s) ds. \tag{5.4}$$

(3) Use a geometric subdivision of $\mathbb{R}^+$, $x_i = r^i$, $i = -n, -n+1, \ldots, 0, 1, \ldots, n-1$ (for some $r \in]1,2[$ and n going to infinity) to discretize the frequency integral (the integral w.r.t. x):

$$\int_0^{+\infty} x^{-\alpha} \psi(x,t,f) dx \# \sum_{i=-n}^{n-1} \int_{x_i}^{x_{i+1}} x^{-\alpha} \psi(x,t,f) dx.$$

Then, by considering the probability density function over the interval $[x_i, x_{i+1}]$ defined by:

$$g_i(x) = \frac{x^{-\alpha}}{c_i}, \quad c_i = \int_{x_i}^{x_{i+1}} x^{-\alpha} dx = \frac{(r^{1-\alpha}-1)}{1-\alpha} r^{(1-\alpha)i} \tag{5.5}$$

and the mean value η_i over this interval for this density function:

$$\eta_i = \frac{1}{c_i} \int_{x_i}^{x_{i+1}} x^{1-\alpha} dx = \frac{1-\alpha}{2-\alpha} \frac{r^{2-\alpha}-1}{r^{1-\alpha}-1} r^i, \tag{5.6}$$

we use the mean value approximation for the frequency integral of interest:

$$\int_{x_i}^{x_{i+1}} x^{-\alpha} \psi(x,t,f) dx \# c_i \psi(\eta_i, t, f).$$

Then the volatility process on a discrete time grid will be recovered from the path of the process X by:

$$(\sigma^2)^{r,n}(t_j) = \theta + \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{i=-n}^{n-1} c_i \Psi(\eta_i, t_j, X).$$

(4) Use a time discretization to compute recursively $\psi(\eta_i, t_j, X)$ along a discrete time sample t_j, $j = 1, 2, \ldots, N$. In order to do this, it is worth noticing that:

$$\Psi(x, t_{j+1}, f) = e^{-x(t_{j+1}-t_j)}\Psi(x, t_j, f) + \int_{t_j}^{t_{j+1}} e^{-x(t_{j+1}-s)} f(s)ds.$$

This suggests the following recursive discretization:

$$\psi(x, t_{j+1}, f)\# e^{-x\Delta}\Psi(x, t_j, f) + f(t_j)\frac{1-e^{-x\Delta}}{x},$$

written, for sake of simplicity, in the simplest case of regularly spaced observations: $t_{j+1} = t_j + \Delta$, $j = 0, 1, \ldots, N$.

To summarize, from observed values $X(t_j)$, $j = 0, 1, \ldots, N$ of the underlying process X, we recover a time discretization of the function ψ:

$$\begin{aligned} \Psi^\Delta(x, t_{j+1}, f) &= \Psi^\Delta(x, t_j, f)e^{-x\Delta} + f(t_j)\frac{1-e^{-x\Delta}}{x} \\ \Psi^\Delta(x, t_0, f) &= 0. \end{aligned} \tag{5.7}$$

and by plugging this discretization in step (3), a discretization of the volatility process

$$(\sigma^2)^{r,n,\Delta}(t_j) = \theta + \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{i=-n}^{n-1} c_i \Psi^\Delta(\eta_i, t_j, X) \tag{5.8}$$

Then we have the following assessment of the rate of convergence of the discretization (5.8):

Proposition 5.1. *For any $\beta \in]0, \frac{1}{2}[$, the random variable*

$$\sup_{(r,n,\Delta)\in[1,2]\times\mathbf{N}^*\times]0,1]} \frac{1}{(r-1)^2(1+\Delta^\beta) + r^{-\alpha n}} \sup_{j=1,\ldots,N} |(\sigma^2)^{r,n,\Delta}(t_j) - (X-\varpi)_0^{(\alpha)}(t_j)|$$

belongs to $\mathbb{L}^p$ for all $p \geq 1$.

Let us emphasize that such a recursive discretization scheme is a necessary tool if one wants to do statistical inference from a large discrete time sample of observations $S(t_j)$, $j = 0, 1, \ldots, N$. It is well-know indeed that the main problem of fractional calculus is its non-recursive aspects and the fact that a standard discretization would require the use of all the previous points to compute the last one, at each time. Formula (5.8) uses only deterministic stored values η_j, c_j, and recursively computable $\Psi^\Delta(\eta_i, t_j, X - \varpi)$, $X(t_j)$.

5.1.3. *Discretization of the fractional derivation.* For estimation purpose, we shall need approximations of the derivation operator, for which similar schemes can be developed.

Let f be a function Hölder continuous of order β. The operator $f \mapsto f^{(-\alpha)}$ given by (3.1) is now compared to

$$f_0^{(-\alpha)}(t) = \frac{\alpha}{\Gamma(1-\alpha)} \left[\frac{f(t)}{\alpha t^\alpha} + \int_0^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}} ds \right].$$

Then we shall use the approximate derivation to be applied to $Z - Z_0$:

$$Z^{\Delta,-\alpha,r,n}(t_j) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{i=-n}^{n-1} c_i' \Xi^\Delta(\eta_i', t_j, Z - Z_0),$$

where Ξ^Δ is computed by

$$\begin{aligned} \Xi^\Delta(x,0,f) &= 0, \\ \Xi^\Delta(x,t_{j+1},f) &= e^{-x\Delta}\Xi^\Delta(x,t_j,f) + (f(t_{j+1}) - f(t_j)). \end{aligned} \tag{5.9}$$

and c_i' and η_i' are given by

$$c_j' = r^{\alpha j} \frac{r^\alpha - 1}{\alpha}, \quad \eta_j' = r^{(\alpha+1)j} \frac{r^{\alpha+1}-1}{c_j'(\alpha+1)}. \tag{5.10}$$

We can prove then (see Appendix) that there exists a constant C such that for any f Hölder continuous of index $\beta > \alpha$, for $r \in]1,2]$, $n \in \mathbf{N}^*$ and $\Delta \in]0,1[$

(5.11)

$$\sup_{j=0,\ldots,N} \left| \sum_{i=-n}^{n-1} c_i'\Xi^\Delta(\eta_i', t_j, f) - f_0^{(-\alpha)}(t_j) \right| \le C[(1+\Delta^{(\beta+\alpha)/2})(r-1)^2 + r^{-n\min(\alpha,(\beta-\alpha)/2)}].$$

Since f in our problem is $Z_t - Z_0$ (or $\sigma^2(t) - \sigma^2(0)$), we know that the considered paths are $\alpha + 1/2 - \epsilon-$Hölder, for any $\epsilon > 0$ so that $\beta - \alpha = 1/2 - \epsilon > 0$.

5.2. **Statistical strategy.** We describe here a statistical strategy based on observations $\sigma^2(t_j)$, $j = 1, \ldots, N$ of the volatility process obtained from high-frequency data under the assumptions of Proposition 3.2. A similar strategy could easily be settled from observations of the integrated volatility process $\int_{t_j}^{t_{j+1}} \sigma^2(u)du$, as described in Section 3.4. In both cases, a comprehensive asymptotic theory should also take into account the measurement error as in Barndorff-Nielsen and Shephard (2002). This issue is left for future work. For sake of expositional simplicity, we focus here on the case without leverage ($\rho = 0$). The approach could easily be extended to accommodate the estimate of ρ.

1. The mean θ and the fractional integration parameter α of the stationary long memory process $\sigma^2(t)$ can be estimated by standard semiparametric procedures. We choose here to simply use the empirical mean $\hat\theta = \frac{1}{n}\sum_{j=1}^n \sigma^2(t_j)$ and the Geweke and Porter-Hudak (1983) log-periodogram regression estimator $\hat\alpha$ as revisited by Robinson (1996). Let us recall that the idea of log-periodogram regression is based on the equivalent $f(\lambda) \sim C\lambda^{-2\alpha}$ of the spectral density in the neighborhood of zero. For a choice of Fourier frequencies $\lambda_k = 2k\pi/n$, for $k = \ell, \ell+1, \dots, m$ and an estimation of the log-periodogram as

$$I_n(\lambda_k) = \frac{1}{2\pi n}\left|\sum_{j=1}^{n}(\sigma^2(t_j) - \hat{\theta})e^{it_j\lambda_k}\right|^2$$

the approximated regression model

$$\ln(I_n(\lambda_k)) \sim \ln(C) - 2\alpha\ln(\lambda_k), \quad k = \ell, \ell+1, \dots, m$$

provides an OLS estimator $\hat{\alpha}$ of α. According to Robinson (1996), fine tuning of the trimming parameters ℓ and m is required to get good properties of $\hat{\alpha}$. Moreover, it is worthwhile to realize that the standard asymptotic theory for the Geweke and Porter-Hudak estimator has only been developed for Gaussian processes, which is not the case of the volatility process. However, in the same way as Velasco (2000) provides results of consistency of log-periodogram based estimators of the short memory parameters for very general linear processes, we can hope to get by this way a consistent estimator $\hat{\alpha}$ of α. The short scale Monte Carlo study of Section 5.3 gives some support to this claim.

2. We use the estimations $\hat{\theta}$ and $\hat{\alpha}$ and the approximation of the fractional derivation operator described in the previous subsection to get implied values of

$$X(t_j) = (\sigma^2 - \theta)^{(-\alpha)}(t_j).$$

Since the process X follows an affine process

$$dX(t) = -kX(t)dt + \gamma\sqrt{X(t) + \varpi}dW^X(t),$$

it is easy to derive consistent asymptotically normal estimators of the unknown parameters k, ϖ and γ. We choose here to estimate these three parameters by a very simple method of moments based on the following three theoretical moments:

$$m_2 = \mathbb{E}(X^2(t)) = \frac{\gamma^2\varpi}{2k}, \quad c_q = \mathrm{cov}[X(t), X(t+q\Delta)] = \frac{\gamma^2\varpi}{2k}e^{-kq\Delta}, \quad m_3 = \mathbb{E}(X^3(t)) = \frac{\gamma^4\varpi}{2k^2}.$$

By denoting by $\hat{m}_2, \hat{c}_q$ and $\hat{m}_3$ the empirical counterparts of m_2, c_q and m_3 respectively, this gives the following estimators:

$$\hat{\varpi} = \frac{2\hat{m}_2^2}{\hat{m}_3}, \quad \hat{k} = \frac{1}{q\Delta}\ln\left[\frac{\hat{m}_2}{\hat{c}_q}\right] \quad \text{and} \quad \hat{\gamma} = \left(\frac{2\hat{m}_2\hat{k}}{\hat{\varpi}}\right)^{1/2}. \tag{5.12}$$

5.3. **Simulation results.**

The CIR process X has been generated with time intervals Δ using the multidimensional method with dimension $d = 3$. For a choice of d, γ and k, the mean is given by $\varpi = d\gamma^2/(4k)$ and the variance by $c_X(0) = \varpi\gamma^2/(2k) = d\gamma^4/8k^2$. Fractional integration is performed to generate σ^2 with the same step Δ using formula (5.8) applied to $\tilde{\sigma}^2$ centered by its empirical mean (instead of the theoretical mean) with a value of θ that ensures that all paths are positive. Comparison of Figures 1 (a) and (b) illustrates the smoothing properties of fractional integration.

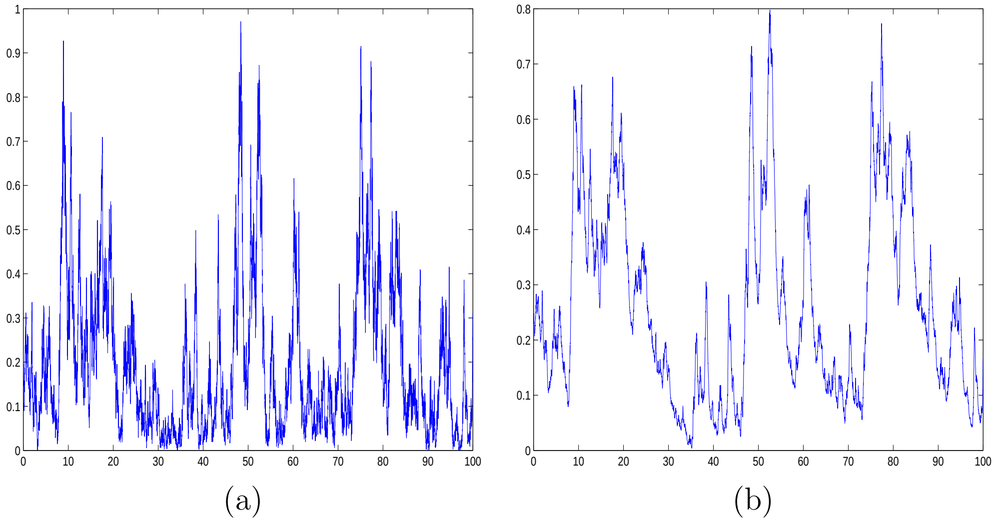

FIGURE 1. (a) A path of CIR generated with the multivariate method ($d = 3$) and parameters $k = 1, \gamma = 0.5, \Delta = 0.01, n = 10000$.
(b) Simulated volatility by fractional integration with $\alpha = 0.25$ of the above path, $\theta = 0.2$.

The discretized fractional integration operator has been tested on deterministic functions and appears to crucially depend on the choice of r (the step of the geometric subdivision): when the sample is small with large step, r must be small (around 1.02-1.04) and for large samples with smaller steps, r must be larger (around 1.3). For a well chosen r, it performs better than the naive method. Besides, we tested the discretized fractional derivation by checking that when applied to a fractionally integrated process, it delivers the original process. Here again, the choice of r must be done carefully. But in any case, the recursive method is considerably faster than the naive one.

The paths of σ^2 are used to construct the paths of the log prices $p(t) = \ln S(t)$ with constant mean $\mu = 0$ with the recursion:

$$p^{\Delta}((k+1)\Delta) = p^{\Delta}(k\Delta) + \sqrt{\sigma^2(k\Delta)}\sqrt{\Delta}\varepsilon_{(k+1)\Delta}$$

where $\sqrt{\Delta}\varepsilon_{(k+1)\Delta} \overset{\mathcal{L}}{=} W^S((k+1)\Delta) - W^S(k\Delta)$ are i.i.d. $\mathcal{N}(0, \Delta)$. Figure 2 plots the resulting paths computed with σ^2 given in Figure 1 (b).

Next, the method of estimating the volatility or the integrated volatility by using sums by block of squared increments of the log-prices is studied. More precisely, Figure 3 (a) shows the difference between the true volatility $\sigma^2(k\Delta)$, for $k = 1, \dots, n$ and the values

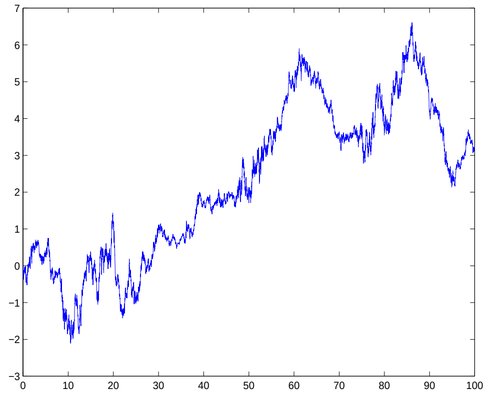

FIGURE 2. Simulated log-prices with the above volatility and mean $\mu = 0$.

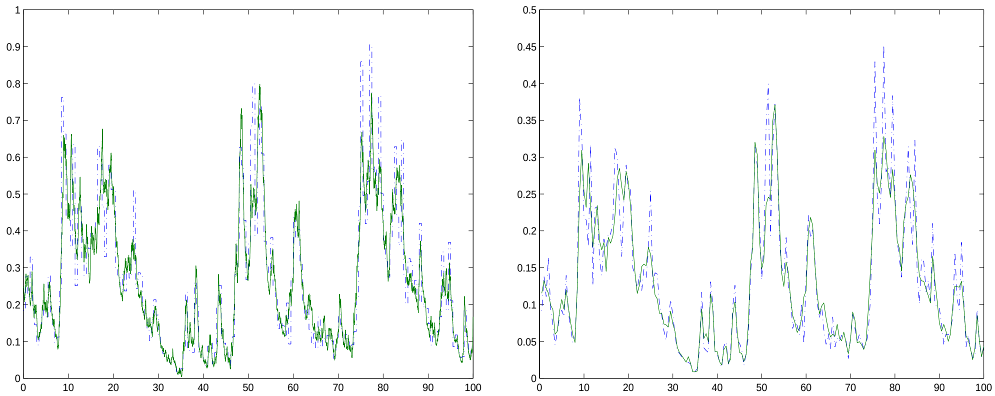

FIGURE 3. (a) Estimated volatility using blocks of size $p = 50$ (dotted line) compared with the true volatility (full line).
(b) Estimated integrated volatility using blocks of size $p = 50$ (dotted line) compared with the true (cumulated) volatility (full line).

for $t = \Delta, 2\Delta, \ldots, n\Delta$

$$\hat{\sigma}^2_{n,p}(t) \;=\; \frac{1}{p\Delta} \sum_{j=([t/(p\Delta)]-1)p+1}^{[t/(p\Delta)]p} \left[R^{\Delta}(j\Delta) - R((j-1)\Delta)\right]^2$$

for $k = 1, \ldots, n/p$, with $p = 50$ and $n = 10000$.

Figure 3 (b) shows the difference between the following approximation of the integrated volatility: $\sum_{j=(k-1)p+1}^{kp} \sigma^2(j\Delta)$ for $k = 1, \ldots n/p$ and the approximation of the quadratic variation $\sum_{j=(k-1)p+1}^{kp} \left[p^{\Delta}((j+1)\Delta) - p^{\Delta}(j\Delta)\right]^2$ for $k = 1, \ldots, n/p$, still with $p = 50$ and $n = 10000$ and on the same path.

In both cases, further simulations show that a smaller value of p ($p = 5, 10, 15$) would produce a too noisy measurement volatility. We also applied the fractional derivation of order α to the estimated volatility process and compared the paths to the initial paths of the CIR (with the same step). The curves coincide quite well and the results are very convincing.

To test the robustness of the log-periodogram regression to non-gaussianity, we estimated α from the complete samples of simulated volatilities using this method. We plot in Figure 4 the histogram of the estimated values of α for 100 samples of length 10 000. The results are quite convincing but the method is very sensitive to the choice of the trimming parameter m. We chose $m/n = 36.35\%$ while the $\ell = 1806$ first terms of all samples have been dropped out. This experiment allows to conclude that the procedure is robust to our case of non-gaussianity.

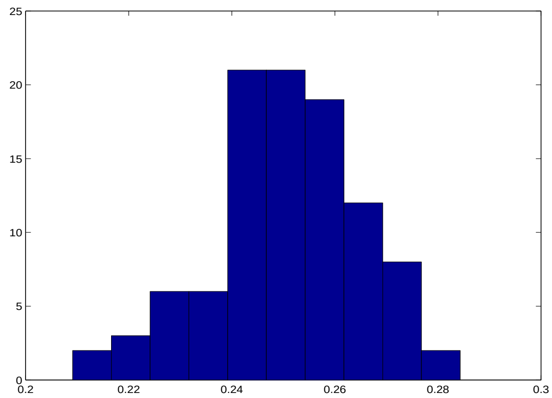

FIGURE 4. Histogram of estimated α over 100 samples of length $n = 10000$ with step $\Delta = 0.04$ for a true value $\alpha = 0.25$, $(k, \gamma, \theta, \varpi) = (1, 0.5, 0.3, 0.3125)$, $d = 5$, $r = 1.3$. Mean = 0.2506, Standard deviation = 0.015.

The results of the global procedure are reported in Table 1 below for $n = 10000$ observations corresponding to 400 days of observations with 25 observations per day. Here, the CIR is generated with $d = 5$, discretized fractional integration is done with $r = 1.3$, quadratic variations are computed using blocks of size 25 to estimate the centered volatility (and not integrated volatility), the log-periodogram regression uses the complete samples with $m = 0.39$ times the number of observations (about 400 remaining), and discretized fractional derivation takes $r = 1.025$.

The results are satisfactory even though the true unknown values of parameters define a volatility process with less mean reversion (smaller k) and more volatility (higher γ). This kind of finite sample bias is common for persistent time series (see e.g. Pastorello et

Parameter	θ	α	γ	k	ϖ
True Value	0.3	0.25	0.5	1	0.3125
Mean	0.304	0.252	0.318	1.227	0.3240
Standard Deviation	0.013	0.059	0.080	0.430	0.242

TABLE 1. Results of the estimation of the parameters of equations n =10 000 and time interval $\Delta = 0.04$.

al. (2000)). Moreover, the rather high values of standard errors could easily be reduced by using a larger set of moment conditions than the minimum one (5.12).

While this Monte Carlo experiment has been performed with high mean level of volatility (around 30 percents), it is worth considering also a case of a volatility process with more mean reversion, less instantaneous variance and a smaller mean level. The true unknown values considered in Table 2 below were provided by an empirical study of Moreaux et al. (1998).

With these values, we first simulate 100 paths of the underlying short memory volatility processes X of length 10000 with step 0.04. We notice that in this low variance case, the size of the blocks used for the quadratic variation procedure had to be chosen much smaller than in the previous higher variance case: namely, we took blocks of size 4 (instead of 25 before). This implies longer samples with smaller step than previously, so that the choice of r in the fractional derivation must be slightly increased to $r = 1.035$. We tested the moment method (see formula (5.12) on the simulated path of X and found perfect results for $q = 1$ given in Table 2.

Parameter	γ	k	ϖ
True Value	0.85	15	0.06
Mean	0.850	15.025	0.060
Standard Deviation	0.019	0.471	0.002

TABLE 2. Results of the estimation of the parameters of the CIR using the (unobservable) 100 paths with length 10 000 and step 0.04, of X generated by the multivariate procedure with $d = 5$.

After application of (2.5) on the 100 simulated paths, all the computed values of $\sigma^2(t)$ appear to be positive. The global procedure gives the results stated in Table 3. In other words, the global procedure performs quite well, and testing simulated data nearer of real data leads to some practical recommendations on the size of blocks and the choice of r.

Parameter	θ	α	γ	k	ϖ
True Value	0.06	0.25	0.85	15	0.06
Mean	0.06	0.252	0.997	14.513	0.048
Standard Deviation	0.049	0.020	0.083	1.470	0.005

Table 3. Results of the estimation of the parameters over 100 samples with length 10 000 and step 0.04.

6. Conclusion

While fractionally integrated stochastic volatility models are well suited at capturing apparent long-run dependencies in the volatility, a continuous time model is relevant to deal with option prices based measures of volatility. A class of fractionally integrated continuous time processes has been introduced by Comte and Renault (1996) and already applied to volatility modelling in Comte and Renault (1998). However, the Comte and Renault (1998) approach of fractional integration of a log-normal volatility process does not allow to clearly disentangle short and long memory properties in the resulting option prices.

By avoiding the non-linearity of a log transformation, direct fractional integration of a square root volatility process allows us to propose in this paper a convenient long memory extension of the Heston (1993) option pricing model. This new option pricing model with stochastic volatility provides a theoretical basis to recent empirical findings as documented in Bandi and Perron (2006). More precisely, the widely spread evidence of unbiasedness of implied volatility as a predictor of realized volatility must be interpreted in terms of presence of a fractional common trend between volatility series. Besides this, the properties of the volatility smile, both in cross section and in term structure are short term variations around the long run common trend.

This disentangling property is crucial to make option pricing with long memory in volatility a feasible task. More precisely, it means that once the relevant long memory features have been captured by computing the value of one given implied volatility in some long term option price, all the additional information needed for option pricing and hedging remains true to a Markov property. In order to improve feasibility, we also provide a way to recursively discretize the fractional integrals so that we can easily recover the underlying short run volatility from observed realized volatility. Moreover, the mere fact that short term and long term properties appear now to be disentangled paves the way for adding any modelling block, like for instance jumps or some transitory volatility factors, to better address the short term option pricing puzzles. The model proposed here only aims at solving the long term issue without compromising on short term properties. In this respect, we have shown that in a pure stochastic volatility model without jumps, short term volatility smiles are not produced by excess kurtosis, as often claimed, but only by the unpredictable part of integrated volatility, that is to say of future realized volatility. When, in the very short term, unpredictability of volatility becomes negligible, volatility smiles tend to flatten themselves while excess kurtosis remains. On the other side of the term structure, our model explains why volatility smiles may remain quite steep, even in the very long term, as observed by Bollerslev and Mikkelsen (1999).

7. Appendix: Proofs

Proof of Proposition 2.1. Let $c_X(h) = \mathrm{cov}(X(t), X(t+h))$. Observe that

$$\|\Delta\|_2^2 := \left\| X^{(\alpha)}(t) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} X(s) ds \right\|_2^2 = \int_{-\infty}^0 \int_{-\infty}^0 \frac{(t-s)^{\alpha-1}(t-u)^{\alpha-1}}{\Gamma(\alpha)^2} c_X(|s-u|) dsdu.$$

Setting $s = tx'$ and $u = ty'$ and then $x = 1 - x', y = 1 - y'$ gives with c_X given by (2.3) that

$$\begin{aligned} \|\Delta\|_2^2 &\leq \frac{t^{2\alpha} c_X(0)}{\Gamma(\alpha)^2} \int_1^{+\infty} \int_1^{+\infty} x^{\alpha-1} y^{\alpha-1} e^{-kt|x-y|} dxdy \\ &\leq \frac{2t^{2\alpha} c_X(0)}{\Gamma(\alpha)^2} \int_1^{+\infty} x^{\alpha-1} e^{ktx} \left(\int_x^{+\infty} y^{\alpha-1} e^{-kty} dy \right) dx. \end{aligned}$$

One integration by parts shows that the integrals is of order $1/t$. □

Proof of Proposition 2.2, 2.3 and 2.4. First note that

$$C(\alpha) := \int_0^{+\infty} u^{\alpha-1}(u+1)^{\alpha-1} du = \int_0^1 x^{-2\alpha}(1-x)^{\alpha-1} dx = \frac{\Gamma(1-2\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha)} \tag{7.1}$$

using a well known formula linking the $\beta(a, b)$ function and the Γ function. Second, we prove that

$$c_{\sigma^2}(h) := \mathrm{cov}(\sigma^2(t), \sigma^2(t+h)) = \frac{\varpi\gamma^2}{2k} \frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_{-\infty}^{\infty} |h+z|^{2\alpha-1} e^{-k|z|} dz. \tag{7.2}$$

Setting $h = 0$ in (7.2) gives the result of Proposition 2.2 since the last integral is equal to $2\Gamma(2\alpha)/k^{2\alpha}$. To prove (7.2), note that

$$\begin{aligned} c_{\sigma^2}(h) &= \int_{-\infty}^{t+h} \int_{-\infty}^{t} \frac{(t+h-s)^{\alpha-1}(t-u)^{\alpha-1}}{\Gamma(\alpha)^2} \mathrm{cov}(X(u), X(s)) duds \\ &= \int_0^{+\infty} \int_0^{+\infty} \frac{x^{\alpha-1} y^{\alpha-1}}{\Gamma(\alpha)^2} c_X(h+y-x) dxdy. \end{aligned} \tag{7.3}$$

Thus, by (7.3) and (2.3), we get

$$c_{\sigma^2}(h) = \frac{\varpi\gamma^2}{2k\Gamma(\alpha)^2} \int_{-\infty}^{t} \int_{-\infty}^{t+h} (t-u)^{\alpha-1}(t+h-v)^{\alpha-1} e^{-k|u-v|} dudv.$$

Then according to the change of variables $z = u - v$ and $r = t - u$ we obtain

$$c_{\sigma^2}(h) = \frac{\varpi\gamma^2}{2k\Gamma(\alpha)^2} \int_{-\infty}^{+\infty} \left(\int_0^{+\infty} (r)^{\alpha-1}(r+h+z)_+^{\alpha-1} dr \right) e^{-k|z|} dz, \tag{7.4}$$

which, with the change of variable $u = r/|h+z|$ and formula (7.1), gives

$$\int_0^{+\infty} (r)_+^{\alpha-1}(r+h+z)_+^{\alpha-1} dr = |h+z|^{2\alpha-1} \frac{\Gamma(1-2\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha)}. \tag{7.5}$$

Then plugging (7.5) into (7.4) yields (7.2) and thus Proposition 2.2.

To prove Proposition 2.3, it remains to prove that, near 0

$$\int_{-\infty}^{\infty} |h+z|^{2\alpha-1} e^{-k|z|} dz = \int_{-\infty}^{\infty} |z|^{2\alpha-1} e^{-k|z|} dz + \frac{k}{\alpha(2\alpha+1)} |h|^{2\alpha+1} + O(h^2). \tag{7.6}$$

Using the change of variable $x = z + h$ in (7.2), we obtain

$$\int_{-\infty}^{\infty} |h+z|^{2\alpha-1} e^{-k|z|} dz = \int_{-\infty}^{\infty} |x|^{2\alpha-1} e^{-k|x-h|} dx.$$

Splitting the previous integral with respect to the sign of $(x-h)$ we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} |h+z|^{2\alpha-1} e^{-k|z|} dz &= 2\int_{0}^{+\infty} x^{2\alpha-1} e^{-kx} dx \cosh(kh) \\ &\quad + \int_0^h |x|^{2\alpha-1} e^{-k(h-x)} dx - \int_0^h |x|^{2\alpha-1} e^{+k(h-x)} dx.\end{aligned}$$

Using the change of variables $y = \frac{x}{h}$ in the two previous integrals over $[0,h]$ we obtain

$$\int_{-\infty}^{\infty} |h+z|^{2\alpha-1} e^{-k|z|} dz = 2\left[\int_0^{+\infty} x^{2\alpha-1} e^{-kx} dx \cosh(kh) - |h|^{2\alpha} \int_0^1 |y|^{2\alpha-1} \sinh(kh(1-y)) dy\right].$$

Then, formula (7.6) is obtained as a consequence of the Taylor expansion of cosh and sinh and the dominated convergence theorem. This yields Proposition 2.3.
We use the same tools for the order near infinity. It remains to prove that near ∞

$$\int_{-\infty}^{\infty} |h+z|^{2\alpha-1} e^{-k|z|} dz = \frac{2}{k}|h|^{2\alpha-1} + O(|h|^{2\alpha}). \tag{7.7}$$

But (7.7) is a consequence of the Taylor expansion of $|1+\frac{z}{h}|^{\alpha-1}$, the Lebesgue dominated convergence theorem and

$$\int_{-\infty}^{\infty} |h+z|^{2\alpha-1} e^{-k|z|} dz = |h|^{2\alpha-1} \int_{-\infty}^{+\infty} \left(1+\frac{z}{h}\right)^{2\alpha-1} e^{-k|z|} dz.$$

This ends the proof of Proposition 2.4. □

Proof of Proposition 2.5: f_X is easily computed as the Fourier Transform of $c_X(h)$ given by (2.3).

$$\begin{aligned}\int e^{-i\lambda h} c_{\sigma^2}(h) dh &= \int_0^{+\infty}\int_0^{+\infty} \frac{x^{\alpha-1}y^{\alpha-1}}{\Gamma(\alpha)^2}\left(\int e^{-i\lambda h} c_{\tilde{\sigma}^2}(h+y-x) dh\right) dx dy \\ &= \int \frac{e^{-i\lambda x} x^{\alpha-1}}{\Gamma(\alpha)} dx \int \frac{e^{i\lambda y} y^{\alpha-1}}{\Gamma(\alpha)} dy \int e^{-i\lambda v} c_X(v) dv, \quad v = h+y-x \\ &= \left|\int \frac{e^{-i\lambda x} x^{\alpha-1}}{\Gamma(\alpha)} dx\right|^2 f_X(\lambda) = \frac{1}{\lambda^{2\alpha}}\left|\int \frac{e^{-iu} u^{\alpha-1}}{\Gamma(\alpha)} du\right|^2 f_X(\lambda) \\ &= \frac{f_X(\lambda)}{\lambda^{2\alpha}}\end{aligned}$$

since from Samko et al. (1993) p.137 $\int_0^{+\infty} t^{\alpha-1} e^{-zt} dt = \Gamma(\alpha)/z^\alpha$, for $z \neq 0$ and $0 < \alpha < 1$ when $Re(z) = 0$.□

Proof of Proposition 3.2. We start with the case $\mu = 0$ and $\rho = 0$. If $p(t) = p_0(t) = \int_0^t \sigma(u) dW^S(u)$, the proof is given in Comte and Renault (1998), Proposition 5.1. This proof uses the independence of σ and W^S, but it can easily be extended as follows. If $m = [tN/T]$, just write

$$\mathbb{E}\left[\hat{\sigma}^2_{N,p}(t) - \sigma^2(t)\right]^2 \le 2\left(\mathbb{E}\left[\hat{\sigma}^2_{N,p}(t) - \sigma^2(t_m)\right]^2 + \mathbb{E}\left[\sigma^2(t_m) - \sigma^2(t)\right]^2\right).$$

Then the first term can be computed as in Comte and Renault (1998) using only predictability and not independence (by conditioning):

$$\mathbb{E}[\hat{\sigma}^2_{N,p}(t) - \sigma^2(t_m)]^2 = \frac{2N^2}{p^2T^2} \sum_{k=m-p+1}^{m} \int\int_{[t_{k-1},t_k[^2} (c_{\sigma^2}(u-v) - c_{\sigma^2}(0))dudv$$

$$+\frac{N^2}{p^2T^2} \int\int_{[t_{m-p},t_m[^2} (c_{\sigma^2}(u-v) - c_{\sigma^2}(0))dudv - \frac{2N}{pT}\int_{[t_{m-p},t_m[^2} (c_{\sigma^2}(t_{m-p+1}-s) - c_{\sigma^2}(0))ds + 2\frac{\mathbb{E}\sigma^4}{p}.$$

Using that $c_{\sigma^2}(h) - c_{\sigma^2}(0) = Ch^{2\alpha+1} + O(h^2)$ for h small leads to the order $2\mathbb{E}(\sigma^4)/p + K(pT/N)^{2\alpha+1}(1+1/p^{2\alpha+2}) + O((pT/N)^2)$ with K proportional to C and using that $t_k - t_{k-1} = T/N$ and $|t_m - t_{m-p}| = pT/N$. The additional term is

$$\mathbb{E}\left[\sigma^2(t_m) - \sigma^2(t)\right]^2 = 2c_{\sigma^2}(0) - 2c_{\sigma^2}(|t_m - t|)$$

so that as $|t_m - t| \le T/N$, we find that the expectation is less than $C(T/N)^{2\alpha+1}$.
Next if $p(t) = \int_0^t \mu(u)du + p_0(t)$, then we have to bound

$$\mathbb{E}_1 = \mathbb{E}\left[\frac{N}{pT}\sum_{k=t_{m-p+1}}^{t_m}\left(\int_{t_{k-1}}^{t_k}\mu(u)du\right)^2\right]^2$$

and

$$\mathbb{E}_2 = \mathbb{E}\left[\frac{N}{pT}\sum_{k=t_{m-p+1}}^{t_m}(p_0(t_k) - p_0(t_{k-1}))\int_{t_{k-1}}^{t_k}\mu(u)du\right]^2.$$

Using that $\left(\int_{t_{k-1}}^{t_k}\mu(u)du\right)^4 \le (t_k - t_{k-1})^3\int_{t_{k-1}}^{t_k}\mu^4(u)du$, we easily find that

$$\begin{aligned}
\mathbb{E}_1 &= \frac{N^2}{p^2T^2}\sum_{k,l}\mathbb{E}\left[\left(\int_{t_{l-1}}^{t_l}\mu(u)du\right)^2\left(\int_{t_{k-1}}^{t_k}\mu(u)du\right)^2\right] \\
&\le \frac{N^2}{p^2T^2}\sum_{k,l}\mathbb{E}^{1/2}\left[\left(\int_{t_{l-1}}^{t_l}\mu(u)du\right)^4\right]\mathbb{E}^{1/2}\left[\left(\int_{t_{k-1}}^{t_k}\mu(u)du\right)^4\right] \le \frac{MT^2}{N^2}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\mathbb{E}_2 &\le \frac{N^2}{p^2T^2}p\mathbb{E}\left[\sum_k (p_0(t_k) - p_0(t_{k-1}))^2\left(\int_{t_{k-1}}^{t_k}\mu(u)du\right)^2\right] \\
&\le \frac{N^2}{pT^2}\sum_k \mathbb{E}^{1/2}\left[(p_0(t_k) - p_0(t_{k-1}))^4\right]\mathbb{E}^{1/2}\left[\left(\int_{t_{k-1}}^{t_k}\mu(u)du\right)^4\right] \le \frac{T}{N}\sqrt{3M\mathbb{E}(\sigma^4)}
\end{aligned}$$

using that

$$\mathbb{E}\left[\left(\int_{t_{k-1}}^{t_k}\mu(u)du\right)^4\right] \le M(t_k - t_{k-1})^4 = \frac{MT^4}{N^4}$$

and that

$$\mathbb{E}\left[(p_0(t_k) - p_0(t_{k-1}))^4\right] = 3\int\int_{[t_{k-1},t_k]^2}\mathbb{E}(\sigma^2(u)\sigma^2(v))dudv \le 3\frac{T^2}{N^2}\mathbb{E}(\sigma^4).$$

Lastly, we mention that this result is robust to some leverage effect i.e. it still holds even if W^S and $W^{\tilde{\sigma}}$ are not independent, which implies only a slight increase in the constant C". All these results are straightforward consequences of Proposition 2.3. This ends the proof of Proposition 3.2. □

Proof of Propositions 4.1 and 4.2: Let $U_t(h)$ be defined by (4.5). Then from equations (4.1) and (4.2), Proposition 4.1 follows since it is straightforward that:

$$\begin{aligned} U_t(h) &= \frac{1}{\Gamma(\alpha+1)}\int_t^{t+h}(t+h-s)^\alpha X(s)ds - G_\alpha(h)X(t) \\ &= \frac{1}{\Gamma(\alpha+1)}\int_0^h (h-s)^\alpha X(t+s) - e^{-ks}X(t))ds \end{aligned}$$

with $G_\alpha(h)$ given by (4.4), which gives Proposition 4.1.

Next, $V_t(\int_t^{t+s}\sigma^2(s)ds) = \mathbb{E}_t(U_t^2(h))$ and

$$\mathbb{E}_t(U_t^2(h)) = \frac{1}{\Gamma(1+\alpha)^2}\int_0^h\int_0^h \prod_{i=1}^2 (h-x_i)^\alpha \mathbb{E}_t\left(\prod_{i=1}^2\left[X(t+x_i)-e^{-kx_i}X(t)\right]\right)dx_1dx_2.$$

Then the result holds if we prove that

$$\mathbb{E}_t\left(\prod_{i=1}^2\left[X(t+x_i)-e^{-kx_i}X(t)\right]\right) = A_2(x_1,x_2)X(t)+B_2(x_1,x_2). \tag{7.8}$$

If for instance $x_1 \le x_2$, by inserting $\mathbb{E}_{t+x_1}$, the conditional expectation is equal to

$$e^{-k(x_2-x_1)}\mathbb{E}_t\left[\left(X(t+x_1)-e^{-kx_1}X(t)\right)^2\right].$$

This term is equal to

$$e^{-k(x_2-x_1)}\mathbb{E}_t\left[X^2(t+x_1)-e^{-2kx_1}X^2(t)\right].$$

Then using the standard following formula

$$\mathbb{E}_t\left[X^2(t+v)\right] = e^{-2kv}X^2(t)+\gamma^2\frac{e^{-kv}-e^{-2kv}}{k}X(t)+\gamma^2\varpi\frac{1-e^{-2kv}}{2k} \tag{7.9}$$

gives

$$e^{-k(x_2-x_1)}\left(\gamma^2\frac{e^{-kx_1}-e^{-2kx_1}}{k}X(t)+\gamma^2\varpi\frac{1-e^{-2kx_1}}{2k}\right).$$

We get

$$A_2(x_1,x_2) = \frac{\gamma^2}{k}e^{-k(x_2-x_1)}(e^{-kx_1}-e^{-2kx_1})$$

and

$$B_2(x_1,x_2) = \frac{\gamma^2\varpi}{2k}e^{-k(x_2-x_1)}(1-e^{-2kx_1}).$$

This gives the result (7.8) for $k=2$, with A and B as defined in Proposition 4.2. □

Proof of Proposition 4.3: It follows from (4.9) that

$$K(h) = 3\left[1+\frac{1}{h^2\mathbb{E}^2(\sigma^2)}\int_0^h\int_0^h c_{\sigma^2}(u-v)dudv\right].$$

From Proposition 2.3, for h small,

$$K(h) = 3\left[1+\frac{1}{h^2\mathbb{E}^2(\sigma^2)}\int_0^h\int_0^h c_{\sigma^2}(0)(1-\kappa(u-v)^{2\alpha+1}+o((u-v)^2))dudv\right],$$

where $\kappa = k^{2\alpha+1}/[(2\alpha+1)\Gamma(2\alpha+1)]$. This leads, for h small, to

$$K(h) = 3\left[1+\frac{1}{\mathbb{E}^2(\sigma^2)}c_{\sigma^2}(0)[1-2\kappa h^{2\alpha+1}/[(2\alpha+2)(2\alpha+3)]+o(h^2)\right].$$

This implies (i).
For h tending to infinity, write

$$K(h)=3\left[1+\frac{2}{h^2\mathbb{E}^2(\sigma^2)}\int_0^h (h-x)c_{\sigma^2}(x)dx\right].$$

From the second part of Proposition, 2.3, it follows that $\int_0^h (h-x)c_{\sigma^2}(x)dx = O(h^{2\alpha+1})$ and this implies (ii). □

Proof of Proposition 4.4: It follows from the definition of Z that

$$cov(Z(t),Z(t+h))=\int_0^1\int_h^{h+1} c_{\sigma^2}(u-v)dudv.$$

The Fourier transform of the previous equation yields:

$$f_Z(\lambda)=\int_{\mathbf{R}} e^{i\lambda h}\int_0^1\int_h^{h+1} c_{\sigma^2}(u-v)dudvdh.$$

Then we use Fubini's Theorem

$$\begin{aligned} f_Z(\lambda) &= \int_0^1\int_{\mathbb{R}} c_{\sigma^2}(u-v)dudv\int_{u-1}^{u} e^{i\lambda h}dh = \frac{1-e^{-i\lambda}}{\lambda}\int_0^1\int_{\mathbb{R}} c_{\sigma^2}(u-v)e^{i\lambda u}dudv \\ &= \frac{1-e^{-i\lambda}}{\lambda}\int_0^1 e^{i\lambda v}dv f_{\sigma^2}(\lambda). \end{aligned}$$

The result follows then from Proposition 2.5 which implies $f_{\sigma^2}(\lambda)=f_{\tilde{\sigma}^2}(\lambda)\lambda^{-2\alpha}$. □

Proof of Proposition 4.5. • The first result is proved in Proposition 2.3.
• Let $u_t=\mathbb{E}_t\sigma^2(t+1)$. Since we can invert the conditional expectation and the integral

$$u_t=\int_{-\infty}^{t+1}\frac{(t+1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathbb{E}_t(X(s))ds+\theta.$$

According to Proposition 2.1, we have

$$\begin{aligned} &\Gamma(\alpha)^2 cov(u_t,u_{t+h}) \\ &= \int_{-\infty}^{t+1}\int_{-\infty}^{t+1+h}(t+1-s)^{\alpha-1}(t+1+h-r)^{\alpha-1}\mathbb{E}[\mathbb{E}_t(X(s))\mathbb{E}_{t+h}(X(r))]dsdr. \end{aligned} \tag{7.10}$$

Observe that, using formula (2.3), if $s\notin[t,t+1]$ or if $r\notin[t+h,t+1+h]$, then

$$\mathbb{E}[\mathbb{E}_t(X(s))\mathbb{E}_{t+h}(X(r))]=c_X(s-r)=\frac{\varpi\gamma^2}{2k}e^{-k|r-s|}, \tag{7.11}$$

whereas if $(s,r)\in[t,t+1]\times[t+h,t+1+h]$

$$\mathbb{E}[\mathbb{E}_t(X(s))\mathbb{E}_{t+h}(X(r))]=\frac{\varpi\gamma^2}{2k}e^{-k(s+r-2t)}. \tag{7.12}$$

Plugging formula (7.11) and (7.12) into (7.10) yields

$$\begin{aligned} cov(u_t,u_{t+h}) &= c_{\sigma^2}(h)+\frac{\varpi\gamma^2}{2k\Gamma(\alpha)^2}\int_t^{t+1}\int_{t+h}^{t+1+h}(t+1-s)^{\alpha-1}(t+1+h-s)^{\alpha-1}e^{-k(r+s-2t)}drds \\ &\quad -\frac{\varpi\gamma^2}{2k\Gamma(\alpha)^2}\int_t^{t+1}\int_{t+h}^{t+1+h}(t+1-s)^{\alpha-1}(t+1+h-s)^{\alpha-1}e^{-k|r-s|}drds. \end{aligned}$$

Then, since $\alpha < \frac{1}{2}$ it remains to prove that for h near ∞

$$M(h) = \int_t^{t+1} \int_t^{t+1+h} (t+1-s)^{\alpha-1}(t+1+h-s)^{\alpha-1} e^{-k(r+s-2t)} drds = O(|h|^{\alpha-1}) \tag{7.13}$$

and

$$N(h) = \int_t^{t+1} \int_t^{t+1+h} (t+1-s)^{\alpha-1}(t+1+h-s)^{\alpha-1} e^{-k|r-s|} drds = O(|h|^{\alpha-1}). \tag{7.14}$$

The change of variables $x = s - t$, $y = r - t - h$ leads to

$$M(h) = e^{-kh} \int_0^1 \int_0^1 (1-x)^{\alpha-1}(1-y)^{\alpha-1} e^{-k(x+y)} dxdy,$$
$$N(h) = \int_0^1 \int_0^1 (1-x)^{\alpha-1}(1-y)^{\alpha-1} e^{-k|-x+y+h|)} dxdy.$$

Taking $h > 1$ yields

$$N(h) = e^{-kh} \int_0^1 \int_0^1 (1-x)^{\alpha-1}(1-y)^{\alpha-1} e^{-k|-x+y|)} dxdy.$$

Therefore, $M(h) = O(e^{-kh})$ and $N(h) = O(e^{-kh})$ when h goes to infinity. The behavior of $\text{cov}(u_t, u_{t+h})$ is given by the one of $c_{\sigma^2}(h)$ which from Proposition 2.3 is given by $c_{\sigma^2}(h) \sim c_{\sigma^2}(0)(kh)^{2\alpha+1}/\Gamma(2\alpha)$. $\square$

• The proof follows the same lines as the previous one. Since we can invert the conditional expectation and the integral, we can write

$$Y(t) = \int_0^1 \int_{-\infty}^{t+u} (t+u-s)^{\alpha-1} \mathbb{E}_t(X(s)) dsdu + \theta.$$

According to Propositions 2.1 and 3.1, we have

$$\begin{aligned}
& \Gamma(\alpha)^2 cov(Y(t), Y(t+h)) \\
= & \int_0^1 \int_0^1 c_{\sigma^2}(h+v-u) dudv \\
& + \int_0^1 \int_0^1 \int_t^{t+u} \int_{t+h}^{t+v+h} (t+u-s)^{\alpha-1}(t+v+h-r)^{\alpha-1} [e^{-k(s-t+r-t)} - e^{-k|r-s|}] dsdrdudv \\
= & \int_0^1 \int_0^1 c_{\sigma^2}(h+v-u) dudv \\
& + \int_0^1 \int_0^1 \int_0^u \int_0^v (u-x)^{\alpha-1}(v-x)^{\alpha-1} [e^{-k(x+y+h)} - e^{-k(|y+h-x|)}] dsdrdudv.
\end{aligned}$$

Clearly for great values of h, the first integral behaves as $c_{\sigma^2}(h)$ and for $h > 1$, using that $k|y + h - x| = k(y + h - x)$ and with the same tools as above, we find that the second integral is of order $O(e^{-kh})$. $\square$

Proof of Proposition 5.1. First, we observe that $(\int_{-\infty}^0 (t-s)^{\alpha-1}[\tilde{\sigma}_c^2(s)]ds,\ t \geq 0)$ goes to 0 when t goes to ∞. Indeed, according to the expression of c_X given in (2.3)

$$\mathbb{E}\left\{\left[\int_{-\infty}^0 (t-s)^{\alpha-1} X(s) ds\right]^2\right\} = \int_{-\infty}^0 \int_{-\infty}^0 (t-s)^{\alpha-1}(t-u)^{\alpha-1} e^{-k|u-s|} duds$$

is dominated by $c_{\sigma^2}(t)$. Then according to Proposition 2.3, we have

$$\mathbf{E}\left\{\left[\int_{-\infty}^0 (t-s)^{\alpha-1} X(s) ds\right]^2\right\} = O(|t|^{2\alpha-1}).$$

As a consequence, the process $X^{(\alpha)}$ to be discretized is approximated by $X_0^{(\alpha)}$ where

$$X_0^{(\alpha)}(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} X(s)ds. \tag{7.15}$$

Second, (5.3) simply follows from

$$(t-s)^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty x^{-\alpha} e^{-x(t-s)} dx,$$

and from Fubini's theorem. We refer to Coutin and Pontier (2007) for further properties of $\Psi(x,t,f)$. Therefore, the discretization is performed as follows, by using representation (5.3). For $r \in [1,2]$ and $n \in \mathbb{N}^*$, let $\pi = (x_i, i = -n, \ldots, n)$ be a geometric subdivision with mesh r and size $2n+1$ given by $x^i = r^i$. To each $i = -n, \ldots, n-1$ we associate c_i and η_i given by (5.5) and (5.6) and the sum

$$I_0^{(\alpha,r,n)}(f)(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{i=-n}^{n-1} c_i \Psi(\eta_i, t, f).$$

Then, according to Lemma 13 of Carmona et al. (2000) applied to $g = \Psi(.,.,f)$, $\mu(dx) = \min(1, x^{-1})$ and $x^{-\alpha}$ and according to the properties of Ψ as given in Theorem 4.10 of Coutin and Pontier (2007), there exists a constant C such that for any f continuous from $[0,T]$ into $\mathbb{R}$

$$\sup_{t\in[0,T]} |I_0^{(\alpha,r,n)}(f)(t) - f_0^{(\alpha)}(t)| \leq C[(r-1)^2 + r^{-n\alpha}].$$

It remains to compute a $\Psi(\eta_i, ., f)$. For that purpose, let $(t_j = j\Delta,\ j = 0, ..., N)$ for $\Delta = \frac{T}{N}$ be a standard subdivision of $[0,T]$. We find the approximation (5.7) and

$$I_0^{(\alpha,r,n,\Delta)}(f)(t_j) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{i=-n}^{n-1} c_i \Psi^\Delta(\eta_i, t_j, f).$$

Applying Corollary 4.2 and Proposition 4.21 of Coutin and Pontier (2007), we obtain the following result:

Proposition 7.1. *There exists a constant C such that for any f β-continuous on $[0,T,]$ then for any $r \in [1,2]$, $n \in \mathbb{N}^*$ and $\Delta \in]0,1]$,*

$$\sup_{j=0,\ldots,N} |\Psi^\Delta(\eta_i, t_j, f) - \Psi(\eta_i, t_j, f))| \leq C\Delta^\beta.$$

Therefore, using Propositions 2.1 and 3.1, we approximate $(\sigma^2 - \varpi)_0^{(\alpha)}$ by (5.8). The result of Proposition 5.1 is then a consequence of Coutin and Pontier (2007).□

Proof of Formula (5.11). First observe that as previously

Lemma 7.1. $(\sigma^2)^{(-\alpha)}(t) - (\sigma^2)_0^{(-\alpha)}(t)$ *goes to 0 when t goes to ∞ in $\mathbb{L}^2(\Omega, \mathbb{P})$.*

Now, it remains to approximate $f_0^{(-\alpha)}(t)$ for any f Hölder continuous of index $\beta > \alpha$. As in the previous section, we write

$$(t-s)^{-(\alpha+1)} = \frac{1}{\Gamma(1+\alpha)} \int_0^{+\infty} x^\alpha e^{-x(t-s)} dx,$$

and we find, using Fubini's Theorem, that for $\alpha \in (0,1)$,

$$f_0^{(-\alpha)}(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^{+\infty} x^{\alpha-1} \left(f(t)e^{-xt} + x \int_0^t e^{-x(t-s)}(f(t) - f(s))ds \right) dx.$$

Then we find the discretization

$$I^{(-\alpha),r,n}(f)(t) = \sum_{j=-n+1}^{n+1} c'_j \Xi(\eta'_j, t, f)$$

where

$$\Xi(x,t,f) = e^{-xt} f(t) + \int_0^t x e^{-x(t-s)}(f(t) - f(s))ds,$$

and c'_j, η'_j are given by (5.10). Then from Proposition 4.1 and Theorem 4.9 of Coutin and Pontier (2007), there exists a constant C such that for any f Hölder continuous of index $\beta > \alpha$,

$$\sup_{t\in[0,T]} |I_0^{(-\alpha)}(f)(t) - I^{(-\alpha),r,n}(f)(t)| \leq C[(r-1)^2 + r^{-n\min(\alpha,\beta-\alpha/2)}].$$

It remains to compute $\Xi(\eta_i, t, f)$. Note that $\Xi(x,t,f) = f(t) - \int_0^t xe^{-x(t-s)} f(s)ds$. For $t_j = j\Delta$, $\Delta = \frac{T}{N}$, we have (5.9). From Coutin and Pontier (2007), we obtain the estimation given by (5.11). □

Proof of Lemma 7.1. We compute

$$R_t = \mathbb{E}\left[\left((\sigma^2)^{(-\alpha)}(t) - (\sigma^2)_0^{(-\alpha)}(t)\right)^2\right] = \frac{\alpha^2}{\Gamma(1-\alpha)^2}\mathbb{E}\left\{\left(\int_{-\infty}^0 \frac{\sigma^2(s)}{(t-s)^{\alpha+1}}ds\right)^2\right\}.$$

Using Fubini's theorem, we obtain

$$R_t = \frac{\alpha^2}{\Gamma(1-\alpha)^2}\int_{-\infty}^0\int_{-\infty}^0 \frac{c_{\sigma^2}(u-s)}{(t-s)^{\alpha+1}(t-u)^{\alpha+1}}duds.$$

This can be written

$$\begin{aligned} R_t &= \frac{\alpha^2}{\Gamma(1-\alpha)^2}\frac{1}{t^{2\alpha}}\left(\int_{-\infty}^0\int_{-\infty}^0 \frac{c_{\sigma^2}(t|x-y|)}{(1-x)^{\alpha+1}(1-y)^{\alpha+1}}dxdy\right) \\ &= \frac{\alpha^2}{\Gamma(1-\alpha)^2}\frac{1}{t^{2\alpha}}\left(\int_1^{+\infty}\int_1^{+\infty} \frac{c_{\sigma^2}(t|x-y|)}{x^{\alpha+1}y^{\alpha+1}}dxdy\right). \end{aligned}$$

Then is clear from formula (7.4) that $|c_{\sigma^2}(h)| \leq c_{\sigma^2}(0)$ for all nonnegative h. Therefore,

$$R_t \leq \frac{\alpha^2}{\Gamma(1-\alpha)^2}\frac{c_{\sigma^2}(0)}{t^{2\alpha}}\left(\int_1^{+\infty}\int_1^{+\infty}\frac{1}{x^{\alpha+1}y^{\alpha+1}}dxdy\right) = O\left(\frac{1}{t^{2\alpha}}\right).$$

It follows that R_t goes to 0 when t goes to ∞ which completes the proof. □

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